Class 1

Real Analysis

Textbook：A. Friedman, Foundations of modern analysis, 1970 (六藝), 1982 (Dover).

References：

1、H.L, Royden, Real analysis, 3rd edition, 1989.

2、R.L. Wheeden & A. Zygmund, Measure and integral, 1977.

3、A. Torchinsky, Real variables, 1988.

Grade：

平時成績：(交習題及點名)

Mid-term：(課堂, open book)　(各$\frac{1}{3}$)

Final：(take home)

課程：

Chap1.Measure theory {Abstract measure theory; Lebesgue measure}
Chap2.Integration; convergence theorems
Chap3.Metric spaces; topology
Arzela-Ascoli, Stone-Weieratrass, Banach fixed point theorems
} real analysis; 1st semester

Chap4.Banach spaces
Chap5.Completely conti. operators
Chap6.Hilbert spaces
} functional analysis; 2nd semester

Mathematical Analysis：

(19th century & before)

Classical analysis：advanced calculus, complex analysis, differential equations

(1) function：differentiation, integration, continuity

(2) sequence of functions：limit

(20th century & after)

Modern analysis：real analysis, functional analysis

(1) vector space of functions
algebra of functions
} algebraic properties.

(2) normed space
topological vector space
} topological properties.

(3) linear functional: duality $\rightarrow$ functional analysis

Comparison of classical & modern analysis:

(1)classical：坐火車自台北到高雄；local, concrete；用顯微鏡

(2)modern：坐飛機自台北到高雄；global, abstract；用望遠鏡

Other integrals：Perron integral,

Daniell integral

Stieltjes integral, etc.

Why measure？

(1)$\therefore$ extension of lengths of intervals.

already in 19th century：Peano, Jordan, Cantor, Borel (only finite additivity)

(2) H. Lebesgue (1875-1941), a pupil of Borel.

(a) Lebesgue measure：countable additivity

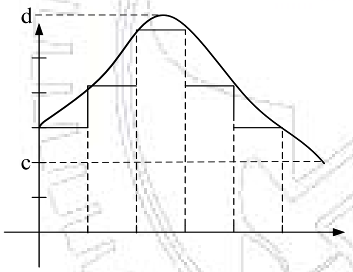
(3) measure the set of discontinuities or the extent of integrability.

(Lebesgue, 1904)

f bdd on $[a,b]$

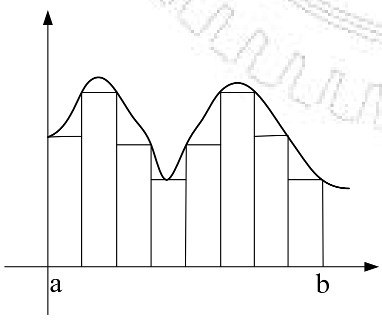
$Then\ f$ Riemann integrable $\Leftrightarrow m\left(\{x\in[a,b]: f \text{ discontinuous at x.}\}\right)=0$

(a) Lebesgue integral



range of function [c,d] divided into subintervals

(b) Riemann integral



domain [a,b] divided into subintervals

Advantage of Lebesgue integral over Riemann integral:

(1) more applicability:

f Riemann integrable $\underset{\not\Leftarrow}{\Rightarrow}$ Lebesgue integrable

Ex. Dirichlet function: $f(x)=\begin{cases}1 & \text{if } x \text{ rational}\\ 0 & \text{if } x \text{ irrational}\end{cases}$ on [0,1]

Then f not Riemann integrable, but Lebesgue integrable.

$$\int_0^1 f(x)dx=0$$

(2) more unified:

(i) domain of integrand:

Riemann integral: finite interval (proper), infinite interval (improper of type Ⅰ)

Lebesgue integral: Lebesgue measurable set E

$$\int_E f(x)dx$$

(ii) range of integrand：

Riemann integral：bdd integrand (proper)

unbdd integrand (improper of type Ⅱ)

Lebesgue integral: arbitrary

(3) simplicity in application:

(i) Convergence:

(a) $f_n \to f$ pointwise on $[a,b]$,

$|f_n| \le M$ $\forall n$ on $[a,b]$

f_n, f Riemann integrable on $[a,b]$

$\Rightarrow \lim \int_a^b f_n = \int_a^b f$ (Arzela's Thm)

(b) $f_n \to f$ pointwise on $[a,b]$

$|f_n| \le M$ $\forall n$ on $[a,b]$

f_n Lebesgue integrable on $[a,b]$

$\Rightarrow f$ Lebesgue integrable & $\lim \int_a^b f_n = \int_a^b f$

(Lebesgues' bdd convergence thm)

Note: 1. Proof of (a) is difficult.

2. f must be assumed integrable.

Ex. Let $\{r_1, r_2, ...\}$ rational no's in [0,1]

Let $f_n(x)=\begin{cases}1 & \text{if } x=r_1,...r_n\\ 0 & \text{otherwise}\end{cases}$ on [0,1]

Then $\int_0^1 f_n(x)dx=0$ $\forall n$

But $f(x)=\begin{cases}1 & \text{if } x \text{ rational}\\ 0 & \text{otherwise}\end{cases}$ on [0,1]

not Riemann integrable

(ii) Fundamental thm. of calculus:

1. Riemann integral:

(a) f Riemann integrable on $[a,b]$.

$F(x)=\int_a^x f(t)dt$ for $x\in[a,b]$

Then f conti. at $x_0\in(a,b)\Rightarrow F'(x_0)$ exists & $F'(x_0)=f(x_0)$.

(b) g' exists on $[a,b]$ & g' Riemann integrable on $[a,b]$.

Then $\int_a^b g'(x)dx = g(b)-g(a)$.

2. Lebesgue integral:

(a) f Lebesgue integrable on $[a,b]$.

$F(x)=\int_a^x f(t)dt$ for $x\in[a,b]$

Then $F'(x)=f(x)$ a.e. on $[a,b]$.

(b) g' exists & bdd on [a,b]

Then g' Lebesgue integrable & $\int_a^b g'(x)dx=g(b)-g(a)$ (cf. p.78, Ex.2.14.7)

Note： f bdd measurable on X, $u(X)<\infty\Rightarrow f$ integrable on X i.e., $L^\infty(X)\subseteq L(X)$

Conclusion: Riemann integral essentially for conti. function.

Lebesgue integral for more general function

Another use:

Probability theory based on real analysis:

Ex. Toss a dice

$\#\{events\}=2^6$

$$\begin{cases}\sigma\text{-algebra} \leftrightarrow \{\text{events}\}\\ \text{probability} \leftrightarrow \text{measure}\\ \text{measurable function} \leftrightarrow \text{random variable} \begin{cases}\text{discrete } r.v.\\ \text{(unif.) conti. } r.v.\end{cases}\\ \text{Radon-Nikodym derivative} \leftrightarrow \text{density function.}\end{cases}$$