Class 11

Chap. 2. Integration

$(X,\ \boldsymbol{a}, u)$ measure space

$X_0 \in \boldsymbol{a}$

Def $f : X_0 \to \mathbb{R}$ measurable if $\forall$ open $M \subseteq \mathbb{R}$, $f^{-1}(M) \in \boldsymbol{a}$.

$f : X_0 \to [-\infty,\infty]$ measurable if $\forall$ open $M \subseteq \mathbb{R}$, $f^{-1}(M) \in \boldsymbol{a}$ & $f^{-1}(\{+\infty\}), f^{-1}(\{-\infty\}) \in \boldsymbol{a}$.

Note. In probability, means. func. = random variable

Thm. $f : X_0 \to \mathbb{R}$. The following are equiv.:

(1) f measurable

(2) $f^{-1}((-\infty,c)) \in \boldsymbol{a}\ \forall c \in \mathbb{R}$;

(3) $f^{-1}((-\infty,c]) \in \boldsymbol{a}\ \forall c \in \mathbb{R}$;

(4) $f^{-1}((c,\infty)) \in \boldsymbol{a}\ \forall c \in \mathbb{R}$;

(5) $f^{-1}([c,\infty)) \in \boldsymbol{a}\ \forall c \in \mathbb{R}$;

(6) $f^{-1}(B) \in \boldsymbol{a}$ $\forall$Borel set $B \subseteq \mathbb{R}$;

Pf. (1) $\Rightarrow$ (2) trivial

(2) $\Rightarrow$ (3)

$$f^{-1}((-\infty,c]) = \bigcap_{n=1}^{\infty} f^{-1}((-\infty, c+\frac{1}{n})) \in \boldsymbol{a}$$

(3) $\Rightarrow$ (4)

$$f^{-1}((c,\infty)) = X_0 \setminus f^{-1}((-\infty,c]) \in \boldsymbol{a}$$

(4) $\Rightarrow$ (5)

$$f^{-1}([c,\infty)) = \bigcap_{n=1}^{\infty} f^{-1}((c-\frac{1}{n},\infty)) \in \boldsymbol{a}$$

(5) $\Rightarrow$ (6)

Let $e = \{A \subseteq \mathbb{R} : f^{-1}(A) \in \boldsymbol{a}\}$

Then e is σ-algebra

(5) says $e \supseteq \{[c,\infty) : c \in \mathbb{R}\}$

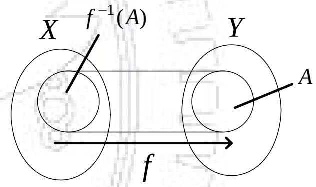
$\Rightarrow e \supseteq \{(-\infty,c) : c \in \mathbb{R}\}$

$\Rightarrow e \supseteq \{[a,b) : a < b \in \mathbb{R}\}$

$\Rightarrow e \supseteq \{(a,b) : a < b \in \mathbb{R}\}$ $(\because (a,b) = \bigcup_{n=1}^{\infty}[a+\frac{1}{n}, b))$

$\Rightarrow e \supseteq \{\text{open sets}\}$

Note:



f^{-1}preserre $\cap,\cup,\setminus$ etc.

Def. $f^{-1}(A) = \{x \in X : f(x) \in A\}$.

inverse image of A under f.

Note 1. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

$f'(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$

Note 2. May consider $f^{-1}(A)$ even if A not invertible.

Ex. $f : \mathbb{R} \to \mathbb{R} \ni f(x) = 0\ \forall x \in \mathbb{R}$.

Then $f^{-1}(A) = \begin{cases} \mathbb{R} & \text{if } 0 \in A \\ \phi & \text{if } 0 \notin A \end{cases}$

$\Rightarrow e \supseteq \{\text{Borel sets}\}$ ($\because e$ σ-algebra)

$\therefore \forall$ Borel set B, $f^{-1}(B) \in \boldsymbol{a}$

(6) $\Rightarrow$ (1): trivial.

(X, ρ) metric space

$X_0 \subseteq X$ open

Def. $f : X_0 \to \mathbb{R}$ conti. if $f^{-1}(O)$ open $\forall$ open $O \subseteq \mathbb{R}$.

Prop. X metric space

u^* metric outer measure

u induced measure

$X_0 \subseteq X$ Borel

$f : X_0 \to \mathbb{R}$ conti. $\Rightarrow f$ measurable on X_0

Pf: $O \subseteq \mathbb{R}$ open

$\Rightarrow f^{-1}(O)$ open in $X_0 \Rightarrow f^{-1}(O)$ Borel in X

$\Rightarrow f^{-1}(O)$ measurable

Note 1. $f :\ X_0 \subseteq \mathbb{R}^n \to \mathbb{R}$ conti. $\Rightarrow$ measurable

2. More generally, upper & lower-semiconti. $\Rightarrow$ measurable (Ex.2.1.11)

Homework: Ex.2.1.8, 2.1.9, 2.1.10

Sec. 2.2. Operations on measurable functions

X, $\boldsymbol{a}$

$f,\ g : X \to [-\infty, \infty]$ measurable

Lma. $f,\ g$ measurable $\Rightarrow \{x \in X : f(x) < g(x)\} \in \boldsymbol{a}$ (also true for " $>$ ", " $\neq$ ", " $=$ ", " $\leq$ ", " $\geq$ ")

Pf. Let $\{r_n\}$ rational no's

$$\| $$

$$\bigcup_n (\{x : f(x) < r_n\} \cap \{x : g(x) > r_n\})$$

$$\| $$

$$\bigcup_n (f^{-1}((-\infty, r_n)) \cap g^{-1}((r_n, \infty))) \in \boldsymbol{a}.$$

Thm. $f,\ g$ measurable, $c \in \mathbb{R}$

Then (1) $f + g$ measurable,

(2) $f - g$ measurable,

(3) $f \cdot g$ measurable,

(4) $\dfrac{f}{g}$ measurable if $g(x) \neq 0\ \forall\ x \in X$

Pf. (1) $\because (f+g)^{-1}((-\infty, c))$

$= \{x : f(x) + g(x) < c\}$

$= \{x : f(x) < c - g(x)\}$

Check: $c - g$ measurable func.

Also, $(f+g)^{-1}(\{\infty\}) = f^{-1}(\{\infty\}) \cup g^{-1}(\{\infty\}) \in \boldsymbol{a}$

$(f+g)^{-1}(\{-\infty\}) = f^{-1}(\{-\infty\}) \cup g^{-1}(\{-\infty\}) \in \boldsymbol{a}$

$\because\ (c-g)^{-1}((-\infty,c_1))$

$=\{x : c-g(x)<c_1\}$

$=\{x : g(x)>c-c_1\}$

$=g^{-1}((c-c_1,\infty))\in\boldsymbol{a}\quad \forall c_1\in\mathbb{R}$

(2) Similar as (1)

(3) "h measurable $\Rightarrow h^2$ measurable" (Ex.2.1.9)

$$\because \left\{x\in X:\ h^2(x)\le c\right\}=\begin{cases}\phi \ \text{ if c} < 0\\ \left\{x\in X:\ h(x)\le\sqrt{c}\right\}\cap\left\{x\in X:\ h(x)\ge -\sqrt{c}\right\}\ \text{if } c\ge 0\end{cases}$$

$\Rightarrow \left\{x\in X:\ h^2(x)\le c\right\}\in\boldsymbol{a}$

$fg=\dfrac{1}{4}((f+g)^2-(f-g)^2)$ measurable.

(4) (Ex.2.2.3) $\because \dfrac{1}{g}$ measurable

$$(\because (\frac{1}{g})^{-1}((-\infty,c))=\begin{cases}g^{-1}(\frac{1}{c},0)\text{ if } c<0\\ g^{-1}(-\infty,0)\text{ if } c=0\\ g^{-1}((-\infty,0]\cup(\frac{1}{c},\infty))\text{ if } c>0\end{cases}\quad\Rightarrow\frac{1}{g}\text{ measurable})$$

Thm. $\{f_n\}$ measurable

$\Rightarrow \sup\limits_n f_n,\ \inf\limits_n f_n,\ \overline{\lim} f_n,\ \underline{\lim} f_n$ measurable.

Pf.: $(\sup\limits_n f_n)^{-1}((-\infty,c])=\left\{x:\ \sup\limits_n f_n(x)\le c\right\}$

$=\bigcap\limits_{n=1}^{\infty}\{x: f_n(x)\le c\}$

$=\bigcap\limits_{n=1}^{\infty} f_n^{-1}((-\infty,c])\in\boldsymbol{a}$

$\inf\limits_n f_n=-\sup\limits_n(-f_n)$ measurable.

$\overline{\lim} f_n=\inf\limits_k\sup\limits_{n\ge k} f_n$ measurable.

$\underline{\lim} f_n=\sup\limits_k\inf\limits_{n\ge k} f_n$ measurable.