Class 12

Note: $f,\ g : X \to [-\infty,\infty]$, u complete measure

$f = g$ a.e. & f measurable $\Rightarrow g$ measurable

$\{x \in X : f(x) \neq g(x)\} \equiv E \in \boldsymbol{a}$ & $u(E)=0$

Pf: Let $E = \{x : f(x) \neq g(x)\}$

Then $u(E) = 0$

$$\begin{array}{ccc} & & \boldsymbol{a}\ (\because f \text{ measurable}) \\ & & \cup \\ & \boldsymbol{a}\ (\because u \text{ complete}) & \{x : f(x) < c\} \cap E^c \\ \because g^{-1}((-\infty,c)) & \cup & \| \end{array}$$

$= \{x : g(x) < c\} = (\underline{\underline{\{x : g(x) < c\} \cap E}}) \cup (\underline{\underline{\{x : g(x) < c\} \cap E^c}})$

Similarly for $g^{-1}(\{\infty\})$ & $g^{-1}(\{-\infty\})$.

Thm. $\{f_n\}$ measurable

(1) $f_n \to g$ pointwise $\Rightarrow g$ measurable.

(2) $f_n \to g$ a.e. & u complete $\Rightarrow g$ measurable.

Pf: (1) $g = \lim_n f_n = \overline{\lim} f_n$ measurable

(2) Let $h(x) = \begin{cases} \lim_n f_n(x) & \text{if conv.} \\ 0 & \text{if div.} \end{cases}$

$E \equiv \left\{x : \lim_n f_n(x) \text{ exists}\right\}$

$= \left\{x : \underline{\lim} f_n(x) = \overline{\lim} f_n(x)\right\}$ is measurable

$\Rightarrow h = (\lim_n f_n)\chi_E$ measurable by Lma 1, below

$h = g$ a.e. $\Rightarrow g$ measurable.

Ex. Dirichlet function: $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ rational} \\ 1 & \text{if } x \text{ irrational} \end{cases}$$

Then $f = 1$ a.e. & f measurable

$(X, \boldsymbol{a})$

Def. $f : X \to \mathbb{R}$ is simple function if $\exists$ disjoint $\{E_1, \ldots E_m\} \subseteq \boldsymbol{a}\ \ni\ E_1 \cup \ldots \cup E_m = X$ & $\alpha_1, \ldots \alpha_m \in \mathbb{R}\ \ni$ $f = d_j$ on $E_j\ \forall j$.

Then $f = \sum_{j=1}^{m} \alpha_j \chi_{E_j}$

Lma. 1. $E \subseteq X$ Then E measurable $\Leftrightarrow \chi_E$ measurable (Ex.2.1.6)

Pf:"$\Rightarrow$" $\because \chi_E^{-1}((-\infty, c)) = \begin{cases} \phi & \text{if } c \le 0 \\ E^c & \text{if } 0 < c \le 1 \\ X & \text{if } c > 1 \end{cases}$

"$\Leftarrow$" $\because E^c = \chi_E^{-1}((-\infty, \frac{1}{2})) \in \boldsymbol{a}$

$\Rightarrow E \in \boldsymbol{a}$

Lma.2. f simple $\Leftrightarrow$ f measurable & $\# f(X) < \infty$

Ex. Let $E \subseteq \mathbb{R}$ not Lebesgue measurable.

Then $f = \chi_E$ not measurable

Thm. $f \ge 0$ measurable

Then $\exists\ f_n$ simple $\ni 0 \le f_n \uparrow f$.

Pf: Let $f_n(x) = \begin{cases} \frac{i-1}{2^n} & \text{if } \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n},\ i = 1,2,..,n2^n. \\ n & \text{if } f(x) \ge n \end{cases}$

$\therefore f_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n_i}} + n\chi_{F_n}$, where $E_{n_i} = f^{-1}([\frac{i-1}{2^n}, \frac{i}{2^n})),\ F_n = f^{-1}([n, \infty])$

$\Rightarrow f_n \ge 0$, simple, $\underbrace{f_n \uparrow}$ & $f_n \le f$.

Check: $f_n(x) \to f(x)\ \ \forall x$.

(1) $f(x) = \infty$

Then $f_n(x) = n \to \infty = f(x)$.

(2) $0 \le f(x) < \infty$: Then $f(x) < n_0$ for some $n_0 \Rightarrow f(x) < n$ for all $n \ge n_0$

$$\Rightarrow 0 \le f(x) - f_n(x) \le \frac{1}{2^n} \quad \forall n \ge n_0$$

$\downarrow$

0 as $n \to \infty$

Cor. f measurable $\Rightarrow \exists\ f_n$ simple $\ni f_n \to f$ pointwise. (Ex.2.2.7)

Pf: $\because f = f^+ - f^-$

$\because \exists\ f_n, g_n$ simple $\ni f_n \to f^+$ & $g_n \to f^-$ pointwise

$\Rightarrow f_n - g_n$ simple & $f_n - g_n \to f$ pointwise.

Homework:Ex.2.2.2 & 2.2.8

$\downarrow$

f bdd measurable $\Rightarrow\ \exists\ f_n$ simple $\ni f_n \to f$ unif. on X

Sec 2.3. Egoroff's Thm.

$f_n \to f$ on X

Convergence: (1) everywhere (pointwise): $\forall\ x \in X,\ f_n(x) \to f(x)$

or $\forall\ x \in X,\ \forall\ \varepsilon > 0\ \exists\ N \ni n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$

(2) almost everywhere: $u(\{x \in X : f_n(x) \not\to f(x)\}) = 0$

(3) unif.: $\forall \varepsilon > 0, \exists N \ni n > N \Rightarrow \sup_{x \in X} |f_n(x) - f(x)| < \varepsilon$

(4) almost unif.

Def. f_n, f measurable on $(X, \boldsymbol{a}, u)$, real-valued a.e.

$f_n \to f$ almost unif. if $\forall \varepsilon > 0, \exists\ E \in \boldsymbol{a} \ni u(\mathrm{E}) < \varepsilon\ \&\ f_n \to f$ unif. on $X \setminus E$.

$\Updownarrow$

$\sup\{|f_n(x) - f(x)|\ x \in X \setminus E.\} \to 0$ as $n \to \infty$

Note: Next thm analogous to "$f_n \to f$ unif. $\Rightarrow f_n \to f$ pointwise"

Thm. $f_n \to f$ almost unif. $\Rightarrow f_n \to f$ a.e.