Class 13

Pf. $\forall m \geq 1, \exists\ E_m \in \boldsymbol{a} \ni u(E_m) < \dfrac{1}{m}$ & $f_n \to f$ unif. on E_m^c.

Let $F = \bigcup_m E_m^c$

Then $f_n \to f$ pointwise on F.

$$u(F^c) = u(\bigcap_m E_m) \leq u(E_m) < \frac{1}{m} \quad \forall m \geq 1 .$$

$\Rightarrow u(F^c) = 0$

$\therefore f_n \to f$ a.e.

Thm. (Egoroff's Thm)

$u(X) < \infty$

$f_n \to f$ a.e. $\Rightarrow f_n \to f$ almost unif.

Pf: Fix $k \geq 1$.

Let $E_n^k = \bigcap_{m=n}^{\infty} \left\{ x \in X : |f_m(x) - f(x)| < \dfrac{1}{k} \right\}$ for $n \geq 1$.

Then $E_n^k \uparrow F \Rightarrow E_n^{k_c} \downarrow F^c$ in n.

$\quad\quad |||$

$\quad \bigcup_n E_n^k$

$\because F \supseteq \{x : f_n(x) \to f(x)\}$

$\Rightarrow F^c \subseteq \{x : f_n(x) \not\to f(x)\}$

$\Rightarrow u(F^c) = 0$

$\because u(E_n^{k_c}) < \infty$

$\Rightarrow u(E_n^{k_c}) \downarrow u(F^c) = 0$

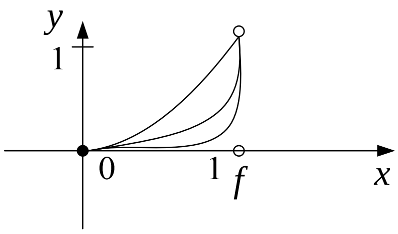
$\therefore \forall \varepsilon > 0$ & $k \geq 1\ \exists\, n_k \ni n \geq n_k \Rightarrow u(E_n^{k_c}) < \dfrac{\varepsilon}{2^k}$.

Let $E = \bigcup_{k=1}^{\infty} E_{n_k}^{k_c} \in \boldsymbol{a}$

(1) $u(E) = u(\bigcup_k E_{n_k}^{k_c}) \leq \sum_k u(E_{n_k}^{k_c}) \leq \sum_k \dfrac{\varepsilon}{2^k} = \varepsilon$

(2) $\forall \delta > 0$, let k be $\ni \dfrac{1}{\mathrm{k}} < \delta$

$\forall x \in E^c = \bigcap_k E_{n_k}^k$



$\because x \in E_{n_k}^k$

$\therefore \left| f_m(x) - f(x) \right| < \dfrac{1}{k} < \delta \quad \forall m \geq n_k$

i.e., $f_n \to f$ unif. on E^c.

Note 1. $f_n \to f$ pointwise on $[a,b] \not\Rightarrow f_n \to f$ unif. on $[a,b]$.

Ex. $f_n(x) = x^n$ on $[0,1]$

Then $f_n(x) \to f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$ pointwrise

But $f_n \not\to f$ unif. on $[0,1]$ $(\because \sup_{x\in[0,1]} \left| f_n(x) - f(x) \right| = 1)$

$f_n \to f$ almost unif.

($\because$ unif. on $[0,1-\varepsilon]$ for any $0 < \varepsilon < 1$)

Note 2. X not finite

Then $f_n \to f$ a.e. $\not\Rightarrow$ $f_n \to f$ almost unif.

(cf. Ex. 2.3.1).

Homework: Ex. 2.3.1