Class 14

Sec. 2.4. Convergence in measure

Def. $\{f_n\}$ a.e. real-valued measurable, f measurable

$f_n \to f$ in measure if $\forall \varepsilon > 0,\ \lim_{n\to\infty} u(\{x : |f_n(x) - f(x)| \ge \varepsilon\}) = 0$

or $\forall \varepsilon > 0,\ \forall \delta > 0,\ \exists N \ni n \ge N \Rightarrow u(\{x : |f_n(x) - f(x)| \ge \varepsilon\}) < \delta$

Properties:

(1) $f_n \to f, g$ in measure $\Rightarrow f = g$ a.e.

Pf: $\underbrace{\{x : |f(x) \ne g(x)|\}}_{E} = \bigcup_m \underbrace{\left\{x : |f(x) - g(x)| \ge \frac{1}{m}\right\}}_{E_m}$

$\therefore E_m \subseteq \left\{x : |f - f_n| \ge \frac{1}{2m}\right\} \cup \left\{x : |f_n - g| \ge \frac{1}{2m}\right\}\ \forall n\ \forall m$

(Reason: $\frac{1}{m} \le |f - g| \le |f - f_n| + |f_n - g| < \frac{1}{2m} + \frac{1}{2m} \quad \to\leftarrow$)

$\Rightarrow u(E_m) = 0\ \ \forall m$

$\Rightarrow u(E) = 0$

(2) $f_n \to f$ in measure $\Rightarrow f$ real-valued a.e.

Pf: Let $E = \bigcup_n \{x : f_n(x) = \pm\infty\}$

Then $u(\mathrm{E}) \le \sum_n u(\{x : f_n(x) = \pm\infty\}) = 0$

Fix $\varepsilon > 0$

$\therefore \{x : f(x) = \pm\infty\} \subseteq ((X \setminus E) \cap \{x : |f_n(x) - f(x)| \ge \varepsilon\}) \cup E\ \ \forall n.$

$\therefore u(\{x : f = \pm\infty\}) \le u(\{x : |f_n - f| \ge \varepsilon\}) \to 0\ \ as\ \ n \to \infty$

$\Rightarrow f$ real-valued a.e.

(3) $f_n \to f$ in measure $\Rightarrow |f_n| \to |f|$ in measure

(4) $f_n \to f$ & $g_n \to g$ in measure, $a, b \in \mathbb{R} \Rightarrow af_n + bg_n \to af + bg$ in measure.

(cf. Ex. 2.4.2 (a) & (b))

(5) $u(X) < \infty$, $f_n \to f$ & $g_n \to g$ in measure $\Rightarrow f_n g_n \to fg$ in measure

(cf. Ex. 2.4.2 (d))

Pf: (i) Check: $\forall \delta > 0,\ \exists c > 0 \ni E \equiv \{x : |g(x)| \le c\} \Rightarrow u(E^c) < \delta$ (almost bdd)

$\forall n$, let $E_n = \{x : |g(x)| \le n\} \in \boldsymbol{a}$

$\therefore E_n \uparrow \bigcup_n E_n$

$\Rightarrow u(E_n) \uparrow u(\bigcup_n E_n) = u(X) < \infty$ ($\because g$ real-valued a.e)

$\therefore \forall\delta>0, \exists c=n \ni u(X)-u(E_n)<\delta$

$\|$

$u(E_n^c)$

Let $E=E_n$

(ii) Check: $f_n g \to fg$ in measure

$\{x:|f_n g - fg|\geq\varepsilon\}\equiv F_n$

$=(F_n\cap E)\cup(F_n\cap E^c)\subseteq\left\{x:|f_n-f|\geq\frac{\varepsilon}{c}\right\}\cup E^c$

$\therefore u(F_n)\leq u(\left\{x:|f_n-f|\geq\frac{\varepsilon}{c}\right\})+u\ (E^c)$

$\wedge$ $\wedge$

δ if n large δ

$\therefore f_n g\to fg$ in measure.

(iii) $f_n\to 0$ in measure $\Rightarrow f_n{}^2\to 0$ in measure.

Pf: $u(\{x:|f_n{}^2|\geq\varepsilon\})=u(\{x:|f_n|\geq\sqrt{\varepsilon}\})\to 0$ as $n\to\infty$

(iv) $f_n\to f$ in measure $\Rightarrow f_n^2\to f^2$ in measure

Pf: $\because f_n - f\to 0$ in measure

(iii) $\Rightarrow (f_n-f)^2\to 0$ in measure

$\|$

$f_n{}^2-2f_n f+f^2$

$+\because 2f_n f\to 2f^2$ in measure (by (ii))

$\Rightarrow f_n{}^2+f^2\to 2f^2$ in measure

$\Rightarrow f_n{}^2\to f^2$ in measure

(v) $f_n\to f$ in measure & $g_n\to g$ in measure $\Rightarrow f_n g_n\to fg$ in measure

Pf: $\because f_n g_n=\frac{1}{4}((f_n+g_n)^2-(f_n-g_n)^2)$

$\to\frac{1}{4}((f+g)^2-(f-g)^2)=fg$ in measure.

(6) $f_n\to f$ in measure, $g_n\to g$ in measure, $g_n, g\neq 0$ a.e. $\forall n$

$\Rightarrow f_n\,|\,g_n\to f\,|\,g$ in measure.

(c.f. Ex. 2.4.2 (e))

Relatoinship between coverges a.e., almost unif. & in measure.

Thm. $f_n\to f$ almost unif. $\Rightarrow f_n\to f$ in measure.

Pf. $\forall\delta>0, \exists E\in\boldsymbol{a} \ni u(E)<\delta$ & $f_n\to f$ unif. on $X\setminus E$.

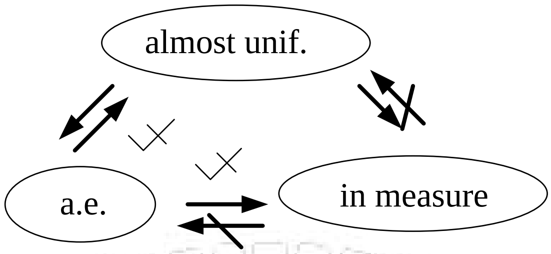
$\Rightarrow\forall\varepsilon>0, \exists N\ni n\geq N\Rightarrow|f_n(x)-f(x)|<\varepsilon \quad \forall x\in X\setminus E$.

$\therefore \{x : |f_n - f| \geq \varepsilon\} \subseteq E, \quad \forall n \geq N$

$\therefore u\ (\{x : |f_n - f| \geq \varepsilon\}) \leq u(E) < \delta \ \ as \ \ n \geq N$

Cor. $u(X) < \infty$, $f_n \to f$ a.e. $\Rightarrow f_n \to f$ in measure.

Pf. By Egoroff & above



Note 1. In general, $f_n \to f$ a.e. $\not\Rightarrow$ $f_n \to f$ in measure.

Ex. $X = \mathbb{R}$

$f_n = \chi_{(n,\infty)}$, $f \equiv 0$.

Then $f_n \to f$ a.e., but $f_n \not\to f$ in measure (Check !)

2. $f_n \to f$ in measure $\not\Rightarrow$ $f_n \to f$ a.e. even if $u(X) < \infty$

cf. Ex. 2.4.5

3. $f_n \to f$ in measure $\not\Rightarrow$ $f_n \to f$ almost unif, even if $u(X) < \infty$.

Reason: by 2.

4. $f_n \to f$ in measure $\Rightarrow f_{n_k} \to f$ almost unif.

Reason: Thm below.

Def. $\{f_n\}$ a.e. real -valued, meas.

$\{f_n\}$ Cauchy in measure if $\forall \varepsilon > 0$, $u(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\}) \to 0$ as $m, n \to \infty$.

or $\forall \varepsilon > 0, \forall \delta > 0, \exists N \ \ni \ m, n \geq N \Rightarrow u(\{x : |f_n - f_m| \geq \varepsilon\}) < \delta$.

Note: $f_n \to f$ in measure $\Rightarrow \{f_n\}$ Cauchy in measure.

Pf: $\{x : |f_n - f_m| \geq \varepsilon\} \subseteq \left\{x : |f_n - f| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x : |f - f_m| \geq \frac{\varepsilon}{2}\right\}$

$(\because \varepsilon \leq |f_n - f_m| \leq |f_n - f| + |f - f_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \to\leftarrow)$