Class 15

Thm. $\{f_n\}$ Cauchy in measure

Then $\exists$ meas. f & f_{n_k} $\ni$ $f_{n_k} \to f$ almost unif.

Pf: $\forall\ k \geq 1$, let $\varepsilon = \delta = \dfrac{1}{2^k}$

$$\Rightarrow \exists n_k \ \ni\ m, n \geq n_k \Rightarrow u(\left\{x : |f_m - f_n| \geq \frac{1}{2^k}\right\}) < \frac{1}{2^k}$$

Assume $n_k \uparrow$

Check: f_{n_k} almost unif conv.

Let $E_k = \left\{x : \left|f_{n_k} - f_{n_{k+1}}\right| < \dfrac{1}{2^k}\right\} \in \boldsymbol{a}$

$F_m = \bigcap\limits_{k=m}^{\infty} E_k \in \boldsymbol{a}$

Note: On F_m, $\left\{f_{n_k}\right\}$ unif. Cauchy.

$\forall x \in F_m,\ h > j > m \Rightarrow x \in E_{h-1}, E_h, \ldots, E_j \Rightarrow \left|f_{n_{h-1}} - f_{n_h}\right| < \dfrac{1}{2^{h-1}}, \ldots, \left|f_{n_j} - f_{n_{j+1}}\right| < \dfrac{1}{2^j}$

$$\Rightarrow \left|f_{n_h} - f_{n_j}\right| < \frac{1}{2^{h-1}} + \ldots + \frac{1}{2^j} \leq \frac{1}{2^j} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{2^{j-1}} < \varepsilon \ \text{ if } j \text{ large.}$$

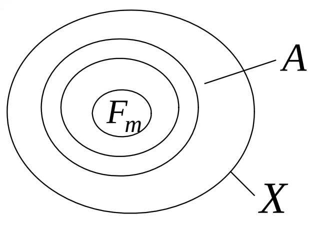
(i.e., $j > \min\{m,\ -\dfrac{\ln\varepsilon}{\ln 2} + 1\}$)

i.e., if h, j large, $\left|f_{n_h} - f_{n_j}\right|$ small $\forall x \in F_m$

$\Rightarrow \left\{f_{n_k}\right\}$ converges unif. to some function on F_m (advanced calculus).

$\because F_m \uparrow$

Let $A = \bigcup\limits_m F_m \in \boldsymbol{a}$



Let $f(x) = \begin{cases} \lim\limits_{k\to\infty} f_{n_k}(x) & \text{if } x \in F_m \\ 0 & \text{if } x \notin A \end{cases}$

Then (1) f measurable. (Reason: $f = \chi_A \cdot \lim\limits_k f_{n_k}$)

(2) $f_{n_k} \to f$ unif. on F_m

(3) $u(F_m^c) \leq \sum\limits_{k=m}^{\infty} u(E_k^c) \leq \sum\limits_{k=m}^{\infty} \dfrac{1}{2^k} = \dfrac{1}{2^m} \cdot \dfrac{1}{1 - \frac{1}{2}} = \dfrac{1}{2^{m-1}}$

i.e., $f_{n_k} \to f$ almost unif.

Cor. 1. $\{f_n\}$ Cauchy in measure $\Rightarrow \exists\ f$ measurable $\ni f_n \to f$ in meas.

Pf: By thm., $\exists$ meas. f, $f_{n_k} \ni f_{n_k} \to f$ almost unif.

$$\Downarrow$$

$$f_{n_k} \to f \text{ in measure.}$$

Note: $\{f_n\}$ Cauchy in measure, $f_{n_k} \to f$ in measure $\Rightarrow f_n \to f$ in meas.

Pf: $\{x : |f_n - f| \geq \varepsilon\} \subseteq \left\{x : |f_n - f_{n_k}| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x : |f_{n_k} - f| \geq \frac{\varepsilon}{2}\right\}$

(Reason: $\varepsilon \leq |f_n - f| \leq |f_n - f_{n_k}| + |f_{n_k} - f|$).

Cor. 2. $f_n \to f$ in measure $\Rightarrow \exists\ f_{n_k} \ni f_{n_k} \to f$ almost unif.

Pf: $\because \{f_n\}$ Cauchy in measure

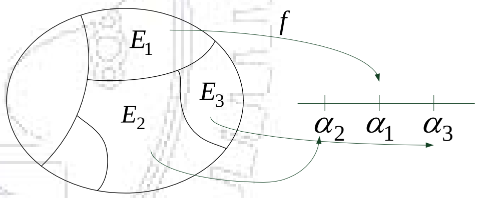
$\therefore$ Thm $\Rightarrow \exists\ f_{n_k}$, $\exists$ g $\ni f_{n_k} \to g$ almost unif.

$$\Downarrow$$

$$\left.\begin{array}{r} f_{n_k} \to g \text{ in measure} \\ \because f_{n_k} \to f \text{ in measure} \end{array}\right\} \Rightarrow f = g \text{ a.e.} \Rightarrow f_{n_k} \to f \text{ almost unif.}$$

Homework: Ex.2.4.2 (a), (b), (e), 2.4.4, 2.4.5

Sec. 2.5 Integrals of simple functions



$(X, \boldsymbol{a}, u)$

$f = \sum_{i=1}^{n} \alpha_i X_{E_i}$ simple $(\{E_i\} \subseteq \boldsymbol{a}$, paritition of X & $\alpha_i \in \mathbb{R})$

$\uparrow$ not necessarily distinct

Def. $\int f du = \sum_{i=1}^{n} \alpha_i u(E_i)$

f integrable if $u(E_i) < \infty\ \forall i$ with $\alpha_i \neq 0$.

Check: $\sum_{i=1}^{n} \alpha_i \chi_{E_i} = \sum_{j=1}^{m} \beta_j \chi_{F_j} \Rightarrow \sum_{i} \alpha_i u(E_i) = \sum_{j=1}^{m} \beta_j u(F_j)$.

Pf: $E_i \cap F_j \neq \phi \Rightarrow \alpha_i = \beta_j \equiv \gamma_{ij}$, say.

$$\therefore \sum_i \alpha_i u(E_i) = \sum_i \alpha_i \sum_{j=1}^{m} u(E_i \cap F_j)$$

$$= \sum_i \sum_j \gamma_{ij} u(E_i \cap F_j)$$

$$\|$$

By symmetry, $\sum_j \beta_j u(F_j)$

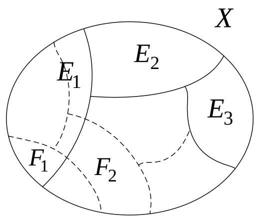
Note.1. f simple, $E \in \boldsymbol{a} \Rightarrow \chi_E f$ simple.

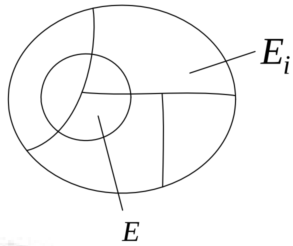
Pf: $f = \sum_{i=1}^{n} \alpha_i X_{E_i} \Rightarrow \chi_E f = \sum_i \alpha_i \chi_{E \cap E_i}$

2. f simple, integrable, $E \in \boldsymbol{a} \Rightarrow \chi_E f$ integrable.

3. f simple, integrable, $E \in \boldsymbol{a}$, $u(E) = 0 \Rightarrow \int_E f du = 0$.

Def. $\int_E f du = \int \chi_E f du$.





Properties:

f , g simple integrable, $\alpha, \beta \in \mathbb{R}$.

(1) $\alpha f + \beta g$ simple integrable & $\int \alpha f + \beta g \, du = \alpha \int f du + \beta \int g du$

Pf: Say, $f = \sum_i \alpha_i \chi_{E_i}$, $g = \sum_j \beta_j X_{F_j}$ where $\cup_i E_i = \cup_j F_j = X, \{E_i\}$ & $\{F_j\}$ disjoint

$\Rightarrow \alpha f + \beta g = \sum_i \alpha \alpha_i \chi_{E_i} + \sum_j \beta \beta_j \chi_{F_j} = \sum_{i,j} (\alpha \alpha_i + \beta \beta_j) \chi_{E_i \cap E_j}$, where $\{E_i \cap E_j\}$ disjoint.

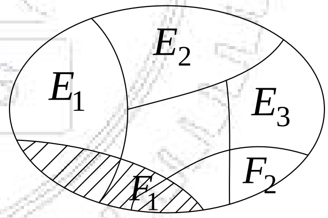
$$\begin{matrix} \| & \| \\ \sum_j \chi_{E_i \cap E_j} & \sum_i \chi_{E_i \cap E_j} \end{matrix}$$

(2) $f \geq 0$ a.e. $\Rightarrow \int f du \geq 0$

Pf : Say, $f = \sum_i \alpha_i \chi_{E_i} \geq 0$ a.e.

$\Rightarrow \alpha_i \geq 0$ for those $i \ni u(E_i) > 0$

$\Rightarrow \int f du = \sum_i \alpha_i u(E_i) \geq 0$.



(3) $f \geq g$ a.e. $\Rightarrow \int f du \geq \int g du$.

Pf : By (1) & (2).

(4) $|f|$ simple integrable & $\left|\int f du\right| \leq \int |f| du$.

Pf: $\because |f| = \sum_i |\alpha_i| \chi_{E_i}$

$$\int f du = \sum_i |\alpha_i| u(E_i) \geq \left| \sum_i \alpha_i u(E_i) \right| = \left| \int f du \right|.$$

(5) $m \le f \le M$ a.e. on $E \in \boldsymbol{a}$, $u(E) < \infty$

$\Rightarrow mu(E) \le \int_E f du \le Mu(E)$

Pf: $\because m\chi_E \le f\chi_E \le M\chi_E$ a.e.

$\uparrow \qquad \uparrow$

simple, integrable

By (3)

(6) $f \ge 0$ a.e., $E \subseteq F \in \boldsymbol{a}$

$\Rightarrow \int_E f du \le \int_F f du$

Pf : $E \subseteq F \Leftrightarrow \chi_E \le \chi_F$

By $\chi_E f \le \chi_F f$ & (3)

(7) $E = \bigcup_m E_m$, $\{E_m\}$ disjoint, $\subseteq \boldsymbol{a}$

$\Rightarrow \int_E f du = \sum_m \int_{E_m} f du \quad \uparrow$

Pf : Say, $f = \sum_i \alpha_i \chi_{F_i}$

LHS $= \sum_i \alpha_i u(E \cap F_i) = \sum_i \alpha_i \sum_m u(E_m \cap F_i) = \sum_m \sum_i \alpha_i u(E_m \cap F_i) = \sum_m \int \chi_{E_m} f du =$ RHS

Note: $E \to \int_E f du$ signed measure

In preparation for the def. of integral:

$\{f_n\}$ integrable, simple functions.

Def. $\{f_n\}$ Cauchy in the mean if $\int |f_n - f_m| du \to 0$ as $n, m \to \infty$.

i.e., $\forall \varepsilon > 0$, $\exists N \ni m, n \ge N \Rightarrow \int |f_n - f_m| du < \varepsilon$

Lma. $\{f_n\}$ integrable, simple, Cauchy in the mean

$\Rightarrow \exists\ f$ a.e. real-valued, measurable $\ni f_n \to f$ in measure.

(In general, $f_n \to f$ in mean $\Rightarrow f_n \to f$ in measure)

Pf: Check: $\{f_n\}$ Cauchy in measure.

$\forall \varepsilon > 0$, $\{x : |f_n(x) - f_m(x)| \ge \varepsilon\} \equiv E_{mn}$.

$\chi_{E_{mn}} |f_n - f_m| \ge \varepsilon \cdot \chi_{E_{mn}} \Rightarrow \int_{E_{mn}} |f_n - f_m| \ge \varepsilon u(E_{mn})$ (By Property (3))

> need $u(E_{mn}) < \infty$
>
> Reason: $\because |f_n - f_m|$ simple $= \sum_{i=1}^{n} \alpha_i \chi_{E_i}$
>
> $\therefore E_{mn} \subseteq \bigcup_{\alpha_i \ne 0} E_i$
>
> $\Rightarrow u(E_{mn}) \le u(\bigcup_{\alpha_i \ne 0} E_i) \le \sum_{\alpha_i \ne 0} u(E_i) < \infty$
>
> $\downarrow$
>
> $\because |f_n - f_m|$ integrable

$\therefore \int |f_n - f_m| \ge \int_{E_{mn}} |f_n - f_m| \ge \varepsilon u(E_{mn})$ if $m, n \to \infty$

Homework: Ex.2.5.2, 2.5.3