Class 16

Sec. 2.6. Integrable functions

$(X, \boldsymbol{a}, u)$

$f : X \to [-\infty, \infty]$ meas.

Def. f integrable if $\exists\ \{f_n\}$ simple,integrable $\ni$

(a) $\{f_n\}$ Cauchy in the mean;
(b) $\lim f_n = f$ a.e.

} note: Both $\int f_n$ & f_n should conv. to $\int f$ & f.

Def. $\int f du = \lim \int f_n du$

Note1. $\lim \int f_n du$ exists, since $\left| \int f_n du - \int f_m du \right| \le \int \left| f_n - f_m \right| \to 0$ as $n, m \to \infty$.

2. $u(X)$ may be ∞ & f may be unbdd $\Rightarrow$ encompassing improper integrals.

3. $f_n \to f$ a.e. $\not\Rightarrow \int f_n \to \int f$

$f_n \to f$ in measure $\not\Rightarrow \int f_n \to \int f$

Thm. f integrable iff $\exists\ \{f_n\}$ simple,integrable $\ni$

(a) & (b′) $f_n \to f$ in meas.

Note 3. From (b′), f integrable $\Rightarrow$ f real a.e.

Pf of Thm. "$\Rightarrow$":

Check: (b′)

Lma & (a) $\Rightarrow$ $f_n \to$ g in meas. $\Rightarrow \{f_n\}$ Cauchy in measure

Thm. $\Rightarrow \exists\ f_{n_k} \to h$ almost unif. $\Rightarrow$ in measure & a.e.

(b) $\Rightarrow$ $f_{n_k} \to f$ a.e. & $f_{n_k} \to g$ in measure

$\therefore$ $f = h$ a.e. & $h = g$ a.e. $\Rightarrow$ $f = g$ a.e.

$\therefore$ $f_n \to f$ in meas.

"$\Leftarrow$":

Check: (a) & (b) for a subsequence f_{n_k}

Thm & (b′) $\Rightarrow \exists\ f_{n_k} \to$ g almost unif.

$\Rightarrow$ $f_{n_k} \to$ g a.e. & $f_{n_k} \to$ g in measure

(b′) $\Rightarrow$ $f_{n_k} \to f$ in measure

$\Rightarrow$ $f =$ g a.e.

$\Rightarrow$ $f_{n_k} \to f$ a.e., i.e, (b) holds

(a) $\Rightarrow \{f_{n_k}\}$ Cauchy in the mean

$\therefore \{f_{n_k}\}$ satisfies (a) & (b).

Thm: f integrable

Then $\int f du = \lim_n \int f_n du$ indep. of $\{f_n\}$ satisfying (a) & (b).

Lma 1. Let f, f_n be as in Def.

Let $\lambda(E) = \lim_n \int_E f_n$ for $E \in \boldsymbol{a}$

Then $\lambda : \boldsymbol{a} \to \mathbb{R}$ is a signed measure.

Motivation:

f integrable on $(X, \boldsymbol{a}, u)$

$\lambda(E) = \int_E f du$ for $E \in \boldsymbol{a}$

Then $\lambda : \boldsymbol{a} \to \mathbb{R}$ is a signed measure.

Check: $\lim_n \int_E f_n$ exists unif. in E.

$\because \left|\int_E f_n - \int_E f_m\right| \le \int_E |f_n - f_m| \le \int |f_n - f_m| \to 0$ as $m, n \to \infty$ by (a)

$\therefore \{\int_E f_n\}$ Cauchy unif. in E.

$\Rightarrow \lim_n \int_E f_n$ exists unif. in E (advanced calculus)

(a) $\lambda(\phi) = \lim_n \int_\phi f_n = \lim_n 0 = 0$

(b) Let $E = \bigcup_i E_i$, $\{E_i\} \subseteq \boldsymbol{a}$, disjoint

Check: $\lambda(E) = \sum_i \lambda(E_i)$

$$\begin{array}{ccc} \| & & \| \\ \lim_n \int_E f_n & & \sum_i \lim_n \int_{E_i} f_n \\ & \| & \text{(by Property (7))} \\ & \lim_n \sum_i \int_{E_i} f_n & \end{array}$$

$\therefore$ To prove two limits (involving n & i) interchangable if one is unif. conti.

Similar to: $f_n \to f$ unif. on E, f_n conti. on $E \Rightarrow f$ conti. on E.

Pf: $|f(y) - f(x)| \le |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \text{ for a fixed large } n \text{ \& } y \to x.$$

$$\therefore \left|\lambda(E)-\sum_{i=1}^{m}\lambda(E_i)\right| \le \left|\lambda(E)-\int_E f_n\right| + \left|\int_E f_n - \sum_{i=1}^{m}\int_{E_i} f_n\right| + \left|\sum_{i=1}^{m}\int_{E_i} f_n - \sum_{i=1}^{m}\lambda(E_i)\right|$$

$$\sum_{i=1}^{m}\int_{E_i} f_n = \int_{\bigcup_{i=1}^{m} E_i} f_n$$

$$\sum_{i=1}^{m}\lambda(E_i) = \sum_{i=1}^{m}\lim_n \int_{E_i} f_n = \lim_n \sum_{i=1}^{m}\int_{E_i} f_n = \lim_n \int_{\bigcup_{i=1}^{m} E_i} f_n = \lambda\left(\bigcup_{i=1}^{m} E_i\right)$$

for fixed f_n, $E \to \int_E f_n$ is countably additive

$\downarrow$

$\le \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}$ for large fixed n & $m \to \infty$

$\uparrow$ $\qquad$ $\uparrow$

$\lambda(E)=\lim_n \int_E f_n$ unif. in $E \Rightarrow \forall \varepsilon>0\ \exists N \ni n>N \Rightarrow \sup_{E\in \boldsymbol{a}} |\lambda(E)-\int_E f_n| < \frac{\varepsilon}{3}$