Class 17

Thm. $f: X \to [-\infty,\infty]$, meas.

$\{f_n\}$, $\{g_n\}$ integrable, simple, Cauchy in the mean,

$\lim_n f_n = \lim_n g_n = f$ a.e.

Then $\lim_n \int f_n = \lim_n \int g_n \ (\equiv \int f)$

Pf: (1) Check: $\lim_n \int_E f_n = \lim_n \int_E g_n$ for $E \in \boldsymbol{a},\ u(E) < \infty$

Note: exist by preceding Lma

$\therefore \left|\lim_n \int_E f_n - \lim_n \int_E g_n\right|$

$= \lim_n \left|\int_E f_n - g_n\right| \le \overline{\lim} \underbrace{\int_E |f_n - g_n|}_{\wedge\backslash}$

$= \int_{E_n} |f_n - g_n| + \int_{E\backslash E_n} |f_n - g_n| \le \int_{E_n} |f_n| + \int_{E_n} |g_n| + \int_{E\backslash E_n} \varepsilon du \ (E_n \equiv \{x \in E : |f_n(x) - g_n(x)| \ge \varepsilon\})$

$\int_{E_n} |f_n|$ $\wedge\backslash$ $\int_{E_n} |f_n - f_N| + \int_{E_n} |f_N|$

$\int_{E\backslash E_n} \varepsilon du$ $\wedge\backslash$ $\varepsilon\mu(E)$

$\int_{E_n} |f_n - f_N|$ $\wedge\backslash$ $\int |f_n - f_N|$ $\wedge$ ε for $n \ge N$ large

$\int_{E_n} |f_N|$ $\wedge\backslash$ $\int_{E_n} c\, du = cu(E_n) < \varepsilon$ for n large

$\because \chi_E(f_n - g_n) \to 0$ a.e. & $u(E) < \infty$

$\Rightarrow \chi_E(f_n - g_n) \to 0$ in measure

$\Rightarrow u(E_n) \to 0$

Similarly for $\int_{E_n} |g_n|$

$\Rightarrow \le 4\varepsilon + \varepsilon\, u(E)$

Let $\varepsilon \to 0$

(2) Check: $\lim_n \int_E f_n = \lim_n \int_E g_n$ for $E \in \boldsymbol{a},\ E = \bigcup_j E_j,\ E_j \in \boldsymbol{a},\ u(E_j) < \infty$

$\because E = \bigcup_j \mathrm{F}_j,\ \mathrm{F}_j \in \boldsymbol{a}$, disjoint & $u(\mathrm{F}_j) < \infty$

$\therefore \text{LHS} = \sum_j \lim_n \int_{F_j} f_n = \sum_j \lim_n \int_{F_j} g_n = \text{RHS}$

↑ Lma ↑ (1) ↑ Lma

(3) Let $N(f_n) = \{x : f_n(x) \ne 0\}$

$N(g_n) = \{x : g_n(x) \ne 0\}$

$N = \bigcup_n [N(f_n) \cup N(g_n)]$

$\because u(N(f_n)), u(N(g_n)) < \infty$ ($\because f_n, g_n$ simple)

(2) $\Rightarrow \lim_n \int_N f_n = \lim_n \int_N g_n$

$\| \quad \| \ (\because \chi_N f_n = f_n \ \ \& \ \ \chi_N g_n = g_n)$

$\int f_n \quad \int g_n$

Note: $E \in \boldsymbol{a}$, f integrable $\Rightarrow \chi_E f$ integrable.

Pf: $\{ f_n \}$ satisfies (a),(b) for f

$\Rightarrow \{ \chi_E f_n \}$ satisfies (a),(b) for $\chi_E f$.

Def. $\int_E f = \int \chi_E f$ if $E \in \boldsymbol{a}$, f integrable.

Special cases:

(1) $\mathbb{R}^n$ with Lebesgue measure

$\int_E f(x)dx$ or $\int f(x)dx$

(2) $\mathbb{R}$ with Lebesgue-Stieltjes measure $u_g((a,b]) = g(b) - g(a)$, where $g \uparrow$, right conti.

$\int_E f dg$, or $\int f dg$.

Note: In one stroke, we have

(1) proper,

(2) improper 1st type

(3) improper 2nd type

(4) multiple

(5) Stieltjes integrals

Homework: Sec. 2.6, Ex. 2.6.2, Ex. 2.6.5, 2.6.6