Class 18

Sec. 2.7 Properties of integrals

Thm f, g integrable, $\alpha, \beta \in \mathbb{R}$

(1) $f = g$ a.e. $\Rightarrow \int f = \int g$

Pf: directly from def.

(2) $\alpha f + \beta g$ integrable & $\int \alpha f + \beta g = \alpha \int f + \beta \int g$.

Pf: $\{f_n\}, \{g_n\}$ integrable, simple, (a), (b) for f, g, resp.

$\Rightarrow \{\alpha f_n + \beta g_n\}$ integrable, simple, (a), (b) for $\alpha f + \beta g$.

$\Rightarrow \alpha f + \beta g$ integrable

$\because \int \alpha f + \beta g = \lim_n \int (\alpha f_n + \beta g_n) = \lim_n (\alpha \int f_n + \beta \int g_n) = \alpha \int f + \beta \int g.$

(3) $f \geq 0$ a.e. $\Rightarrow \int f \geq 0$

Pf: $\{f_n\}$ as above.

$\Rightarrow \{|f_n|\}$ integrable, simple, (a), (b) for $|f|$

$\because f = |f|$ a.e. $\Rightarrow \int f = \int |f| = \lim_n \int |f_n| \geq 0$

(4) $f \geq g$ a.e. $\Rightarrow \int f \geq \int g$

Pf: By (2) & (3)

(5) $|f|$ integrable & $|\int f| \leq \int |f|$ (Ex. 2.6.3)

$\Updownarrow$

f integrable

Pf: "$\Rightarrow$": $\{f_n\}$ as above.

$\Rightarrow \{|f_n|\}$ integrable, simple, (a), (b) for $|f|$.

$\therefore |f|$ integrable.

$\because |\int f_n| \leq \int |f_n|$

$\downarrow \qquad \downarrow$

$|\int f| \qquad \int |f|$

Another proof: f integrable $\Leftrightarrow f^+, f^-$ integrable

$\Leftrightarrow |f|$ integrable. ("$\Leftarrow$" by Thm. 2.10.1)

Easier proof for "$\Leftarrow$": Let $E = \{x : f(x) \geq 0\} \Rightarrow f^+ = \chi_E |f| \Rightarrow f^+$ integrable

Note: (1) For proper Riemann integral, f integrable $\underset{\not\Leftarrow}{\Rightarrow} |f|$ integrable.

Ex. $f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ -1 & \text{if } x \text{ irrational} \end{cases}$ on $[0,1]$.

Then $|f| \equiv 1$

$\therefore \int_0^1 |f| = 1$ but $\int_0^1 f$ not exist. Note: f Lebesgue integrable ($\because f = -1$ a.e.)

(2) For improper Riemann integral,

f integrable $\underset{\not\Rightarrow}{\Leftarrow} |f|$ integrable (as infinite series)

Ex. 1. $f(x)=\dfrac{\sin x}{x}$ on $[1,\infty)$

Then $\int_1^\infty f(x)dx$ conv, but $\int_1^\infty |f(x)|dx$ diverge (i.e., conditionally conv.)

Ex. 2. $f(x)=\dfrac{\sin\frac{1}{x}}{x}$ on $(0,1]$

Then $\int_0^1 f(x)dx$ conv., but $\int_0^1 |f(x)|dx$ diverge (cf. Kirillov & Gvishiani, Prob. 191)

Note: both not Lebesgue integrable

(6) $u(E)<\infty,\ m\le f\le M$ a.e. on $E\Rightarrow mu(E)\le\int_E f\le Mu(E)$

Pf: $\because m\chi_E\le f\chi_E\le M\chi_E$ a.e. $\Rightarrow\int_E m\le\int_E f\le\int_E M$ by (4)

$\uparrow$ (simple), $\uparrow$ (integrable)

simple, integrable

(7) $f\ge 0$ a.e., $E\subseteq F\in\boldsymbol{a}\Rightarrow\int_E f\le\int_F f$

Pf: $\because\chi_E\le\chi_F\Rightarrow\chi_E f\le\chi_F f\Rightarrow\int_E f\le\int_F f$

(8) $m>0$, f integrable, $E=\{x:|f(x)|\ge m\}\Rightarrow u(E)<\infty$

Motivation: $\infty>\int_E|f|\ge\int_E m=mu(E)$; but $u(E)$ not known finite.

Pf: $\because E\subseteq N(f)=\{x: f(x)\ne 0\}$ σ-finite (Ex. 2.6.2)

$\Rightarrow E$ σ-finite

Let $E_j\in\boldsymbol{a}\ \ni\ E_j\uparrow E,\ u(E_j)<\infty\ \forall_j\Rightarrow u(E_j)\uparrow u(E)$

$\infty>\int_E|f|\ge\int_{E_j}|f|\ge mu(E_j)\to mu(E)$

$\uparrow\qquad\uparrow\qquad\uparrow$

$(\because|f|$ integrable, $\chi_E|f|\ge\chi_{E_j}|f|\ge m\cdot\chi_{E_j})$

$\Rightarrow u(E)<\infty$

Note: Trivial for simple integrable f.

Actually, f simple, integrable $\Rightarrow u(\{x\in X: f(x)\ne 0\})<\infty$

Def. $\{f_n\}, f$ integrable (or in L^1)

$f_n\to f$ in mean (or in L^1) if $\int|f_n-f|\to 0$ as $n\to\infty$.

Def. $\{f_n\}$ Cauchy in mean if $\int|f_n-f_m|\to 0$ as $n,m\to\infty$.

Note: $f_n\to f$ in mean $\Leftrightarrow\{f_n\}$ Cauchy in mean

Pf: "$\Rightarrow$" easy.

"$\Leftarrow$" difficult; cf. Sec. 2.8.

Thm. $\{f_n\}$, f integrable

$f_n \to f$ in mean $\Rightarrow f_n \to f$ in measure.

Pf: $\forall \varepsilon > 0$, let $E_n = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$

By (g) above, $u(E_n) < \infty$

$$\searrow$$

$$\therefore \int |f_n - f_m| \geq \int_{E_n} |f_n - f| \geq \int_{E_n} \varepsilon = \varepsilon u(E_n)$$

$$\downarrow$$

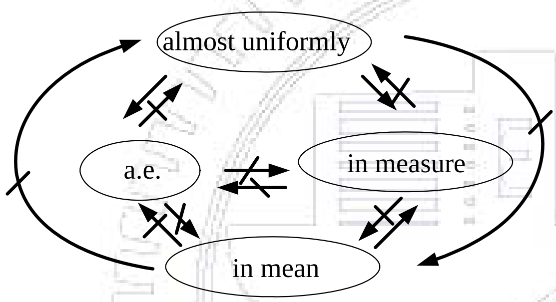
$$0$$

$\Rightarrow u(E_n) \to 0$ as $n \to \infty$.

Note: Similar to Lma 2.5.2.

Note: Convergence in measure: domain small: $\vdash\!\dashv \to 0$

Convergence in mean: area small: ■ $\to 0$

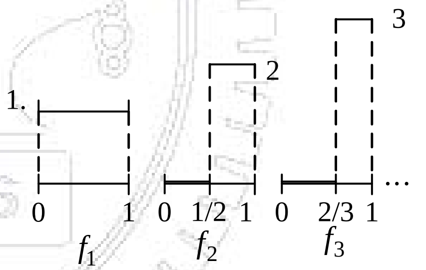


Ex. 1. in measure, almost unif, a.e. $\not\Rightarrow$ in mean.

Then $f_n \to 0$ a.e., but not in mean.

& $f_n \to 0$ almost unif.

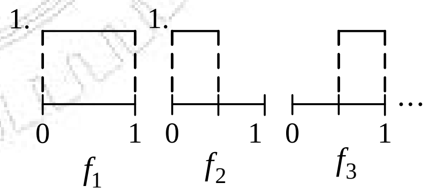
& $f_n \to 0$ in measure.



Ex.2. in mean $\not\Rightarrow$ a.e., almost unif.

Then $f_n \to 0$ in mean, but $f_n(x) \not\to 0$ $\forall x \in [0,1]$.

(similar as Ex. 2.4.5)



Thm. $f \geq 0$ a.e., integrable. (cf. Ex. 2.5.3 for simple f)

Then $f = 0$ a.e. $\Leftrightarrow \int f = 0$

Pf: "$\Rightarrow$": $\because f = 0$ a.e. $\Rightarrow \int f = \int 0 = 0$

"$\Leftarrow$": $\because f$ integrable

$\therefore \exists \{f_n\}$ simple, integrable $\ni$ (a) $\{f_n\}$ Cauchy in mean;

(b) $f_n \to f$ in meas.

$\Rightarrow \{|f_n|\}$ simple, integrable $\ni$ (c) $\{|f_n|\}$ Cauchy in mean;

(d) $|f_n| \to |f| = f$ in meas.

$\therefore \int f = \lim_n \int |f_n|$ by def.

$\|$

0

i.e., $f_n \to 0$ in mean.

$\Rightarrow f_n \to 0$ in meas.

(b) $\Rightarrow f = 0$ a.e.

Note: Next two thms are converses to each other.

Thm. f measure, $E \in \boldsymbol{a} \ni u(E) = 0$

$\Rightarrow f$ integrable on E & $\int_E f = 0$

Pf: $\because \chi_E f = 0$ a.e. $\Rightarrow \chi_E f$ integrable & $\int_E f = \int 0 = 0$

Thm. f integrable, $f > 0$ a.e. on $E \in \boldsymbol{a}$.

(or $<$)

If $\int_E f = 0$, then $u(E) = 0$

Pf: Let $E_n = \left\{ x \in E : f(x) \geq \frac{1}{n} \right\}$ for $n \geq 1$

Moreover, $u(E_n) < \infty$ by (8).

$\swarrow$

$\therefore \int_E f \geq \int_{E_n} f \geq \frac{1}{n} u(E_n) \geq 0$

$\|$

0

$\Rightarrow u(E_n) = 0 \quad \forall n$

Then $E_n \uparrow \bigcup_n E_n$

$\therefore u(E_n) \uparrow u(\bigcup_n E_n) = u(E)$

$\Rightarrow u(E) = 0$

Thm. f integrable, $\int_E f = 0 \quad \forall E \in \boldsymbol{a} \Rightarrow f = 0$ a.e.

Pf: Let $E = \{ x : f(x) > 0 \}$

Thm above $\Rightarrow u(E) = 0$

Let $F = \{ x : f(x) < 0 \}$

By above, $u(F) = 0$

$\Rightarrow f = 0$ a.e.

Homework: Ex. 2.7.2, 2.7.3, 2.7.6

Note: Difficult to compute $\int f$; for real-valued f on $\mathbb{R}$, cf. Riemann integral later.

Section 2.8

$\{f_n\}$, f integrable

Def. $f_n \to f$ in mean if $\int |f_n - f| \to 0$ as $n \to \infty$.

$\{f_n\}$ Cauchy in mean if $\int |f_n - f_m| \to 0$ as $n, m \to \infty$.

Note: $f_n \to f$ in mean $\Rightarrow \{f_n\}$ Cauchy in mean.

Thm 1. $\{f_n\}$ integrable, Cauchy in mean $\Rightarrow \exists f$ integrable $\ni f_n \to f$ in mean.

(i.e., L^1 is complete)

Lma. f integrable,

$\{f_n\}$ simple, integrable, Cauchy in mean & $f_n \to f$ a.e.

Then $f_n \to f$ in mean.

Note 1: $f_n \to f$ a.e. $\not\Rightarrow f_n \to f$ in mean.

Note 2: By def., $\int f_n - \int f \to 0$; now stronger: $\int |f_n - f| \to 0$.

Pf: Fix $n \ge 1$, consider $\{|f_n - f_m|\}_m$

Then simple, integrable, Cauchy in mean, & $|f_n - f_m| \to |f_n - f|$ a.e. as $m \to \infty$

$\uparrow$

Reason: $\int \big\| f_n - f_m| - |f_n - f_l| \big| \le \int |f_m - f_l| \to 0$ as $m, l \to \infty$

By def., $\lim\limits_{m} \int |f_n - f_m| = \int |f_n - f|$

$\therefore \lim\limits_{n,m} \int |f_n - f_m| = \lim\limits_{n} \int |f_n - f|$

$\|$

0

i.e., $f_n \to f$ in mean.

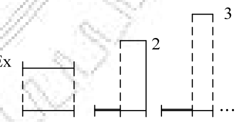
Note 3: In Lma, $f_n \to f$ a.e. replaced by $f_n \to f$ in meas.

Conclusion: f integrable

$\Leftrightarrow \exists f_n$ simple, integrable, $\ni f_n \to f$

a.e., or, in measure, or in mean.

Ex



Note 4: $f_n \to f$ in meas. $\not\Rightarrow f_n \to f$ in mean.

Thm 2. $\{f_n\}$ integrable, Cauchy in mean, & $f_n \to f$ a.e.

Then f integrable, $f_n \to f$ in mean. ($\Rightarrow \lim\limits_{m} \int f_n = \int f$)

Pf: (1) Assume $f_n \to f$ in meas.

Idea: replace each f_n by simple function, by Lma, then use Lma.

For each f_n, by Lma, $\exists$ simple, integrable $\tilde{f}_n \ni \int |\tilde{f}_n - f_n| < \dfrac{1}{n^2}$

$\therefore \{\tilde{f}_n\}$ simple, integrable, Cauchy in mean,

Reason: $\int\left|\tilde{f}_n-\tilde{f}_m\right|\le\int\left|\tilde{f}_n-f_n\right|+\int\left|f_n-f_m\right|+\int\left|f_m-\tilde{f}_m\right|\le\frac{1}{n^2}+\varepsilon+\frac{1}{m^2}$

& $\tilde{f}_n\to f$ in meas.

Reason: Let $E_n=\left\{x:\left|\tilde{f}_n(x)-f_n(x)\right|\ge\frac{1}{n}\right\}$

$\left|\tilde{f}_n-f_n\right|$ integrable $\Rightarrow u(E_n)<\infty$ (by Thm.2.7.1(h))

$\therefore \chi_{E_n}\frac{1}{n}\le\chi_{E_n}\left|\tilde{f}_n-f_n\right|$

$\Rightarrow \frac{1}{n}u(E_n)\le\int_{E_n}\left|\tilde{f}_n-f_n\right|\le\int\left|\tilde{f}_n-f_n\right|<\frac{1}{n^2}$

$\Rightarrow u(E_n)\le\frac{1}{n}\to 0$ as $n\to\infty$

For $\varepsilon>0$, consider $n>\frac{1}{\varepsilon}$.

$\therefore \left\{x:\left|\tilde{f}_n-f_n\right|\ge\varepsilon\right\}\subseteq\left\{x:\left|\tilde{f}_n-f_n\right|\ge\frac{1}{n}\right\}$

$\therefore \mu(\left\{x:\left|\tilde{f}_n-f_n\right|\ge\varepsilon\right\})\le\mu(\left\{x:\left|\tilde{f}_n-f_n\right|\ge\frac{1}{n}\right\})\to 0$ as $n\to\infty$

$\therefore \tilde{f}_n-f_n\to 0$ in meas.

$+\ f_n\to f$ in meas.

$\therefore \tilde{f}_n\to f$ in meas.

$\Rightarrow \exists\,\tilde{f}_{n_k}\ni\tilde{f}_{n_k}\to f$ almost unif. $\Rightarrow \tilde{f}_{n_k}\to f$ a.e.

$\therefore$ By def., f integrable & $\tilde{f}_{n_k}\to f$ in mean by Lma.

$\because \int\left|f_n-f\right|\le\int\left|f_n-\tilde{f}_n\right|+\int\left|\tilde{f}_n-\tilde{f}_{n_k}\right|+\int\left|\tilde{f}_{n_k}-f\right|\le\frac{1}{n^2}+\varepsilon+\varepsilon$

$\Rightarrow f_n\to f$ in mean.

(2) Next assume $f_n\to f$ a.e.

Idea: passing to subsequence & use (1)

$\because\{f_n\}$ Cauchy in mean

$\Rightarrow\{f_n\}$ Cauchy in meas. (proof as conv. in mean $\Rightarrow$ conv. in meas.)