Class 19

Class 20

Def. f integrable on X

$\lambda(E)=\int_E f du \ \forall E\in \boldsymbol{a}$. (indefinite integral of f)

Thm (1) λ is a finite signed meas.

(2) λ is abso. conti. w.r.t. u.

Pf: (1) $\begin{cases}\lambda(\phi)=\int X_\phi f du=\int 0\, du=0\\ \text{Lma. 2.6.3}\Rightarrow \lambda(E)=\int_E f=\lim\limits_m \int_E f_n du \text{ is countably additive.}\end{cases}$

$\therefore \lambda$ signed measure.

Hahn decomposition of X:

Let $A=\{x: f(x)\geq 0\}$

$B=\{x: f(x)<0\}$

Then $A,B\in \boldsymbol{a},\ X=A\cup B,\ A\cap B=\phi$

$A\geq 0,\ B\leq 0$, w.r.t. u.

Jordan decomposition of λ:

$\lambda^+(E)=\lambda(E\cap A)=\int_{E\cap A} f$

$\lambda^-(E)=-\lambda(E\cap B)=-\int_{E\cap B} f$

$\therefore |\lambda|(E)=\int_{E\cap A} f-\int_{E\cap B} f$ finite $\forall E\in \boldsymbol{a}$

$\Rightarrow \lambda$ finite.

(2) Let $\{f_n\}$ simple, integrable, Cauchy in mean & $f_n\to f$ a.e. $\Rightarrow f_n\to f$ in mean.

$|\lambda(E)|=|\int_E f|\leq|\int_E f-\int_E f_n|+|\int_E f_n|$ (Motivation: replace f by bdd f_n).

$\wedge\backslash \qquad \wedge\backslash$

$\int_E|f-f_n| \qquad \int_E|f_n|$ ($\because f_n$ simple, $|f_n|\leq c$, say)

$\wedge\backslash \qquad \wedge\backslash$

$\int|f-f_n| \qquad cu(E)<c\delta\leq\dfrac{\varepsilon}{2}$

$\wedge$

$\dfrac{\varepsilon}{2}$ if n large (Lma.2.8.1) $\forall E\in \boldsymbol{a}$

Let $\delta=\dfrac{\varepsilon}{2}\cdot\dfrac{1}{\text{c}}$

Cor. 1. f integrable

$E,\ E_n\in \boldsymbol{a},\ E_n\to E$ (Analogous: $F(x)=\int_{\text{a}}^x f(t)dt$ conti. in x)

Then $\int_{E_\text{n}} f\to\int_E f$

Pf: $\because \lambda^+, \lambda^-$ finite measures

$$\Rightarrow \lambda^+(E_n) \to \lambda^+(E)$$

$$-\lambda^-(E_n) \to -\lambda^-(E) \quad \text{(Cor. 1.2.3)}$$

$$\lambda(E_n) \to \lambda(E)$$

Cor. 2. f integrable, $E_n \in \boldsymbol{a}$, $\forall n, u(E_n) \to 0$

$\Rightarrow \int_{E_n} f du \to 0$

Pf: $\because \lambda$ abso. conti. w.r.t. u.

$\Rightarrow \forall \varepsilon > 0,\ \exists \delta > 0 \ni u(E) < \delta \Rightarrow |\lambda(E)| < \varepsilon$

$\therefore \forall n > 0,\ \exists N \ni n \ge N \Rightarrow u(E_n) < \delta \Rightarrow \left|\int_{E_n} f du\right| < \varepsilon.$

Def. $g:[a,b] \to \mathbb{R}$ is abso. conti. w.r.t. Lebegue measure if

$\forall \varepsilon > 0, \exists \delta > 0 \ni$ for countably disjoint $(a_i, b_i) \subseteq [a,b]$ with $\sum_i (b_i - a_i) < \delta \Rightarrow \sum_i |g(b_i) - g(a_i)| < \varepsilon$

$f:[a,b] \to \mathbb{R}$

Def: $T_a^b(f) = \sup\left\{\sum_i |f(x_i) - f(x_{i-1})| : a = x_0 < x_1 < \ldots < x_n = b\right\}$

(Total variation of f over $[a,b]$)

Def: f of bdd variation over $[a,b]$ if $T_a^b(f) < \infty$

$BV[a,b] = \{f$ on $[a,b]$ of bdd variation$\}$

Def: $P_a^b(f) = \sup\left\{\sum_i (f(x_i) - f(x_{i-1}))^+ : a = x_0 < x_1 < \ldots < x_n = b\right\}$

(Positive variation of f over $[a,b]$)

$N_a^b(f) = \sup\left\{\sum_i (f(x_i) - f(x_{i-1}))^- : a = x_0 < x_1 < \ldots < x_n = b\right\}$

Neg. variation of f over $[a,b]$.

Note: Similarly, $[a,b] \to \mathbb{R}^2 : t \mapsto f(t) = (x(t), r(t))$

Then $T_a^b(f) =$ curve length; curve rectifiable if $T_a^b(f) < \infty$

Properties: (cf. Royden, Chap. 5. Sec. 2)

Note: $x \in \mathbb{R},\ \frac{1}{2}(x + |x|) = x^+ = \max\{x, 0\}$

$$x^- = -\min\{x, 0\} = \frac{1}{2}(|x| - x)$$

$$\therefore\ x = x^+ - x^-;\ |x| = x^+ + x^-$$

(1) $f \uparrow$ on $[a,b]$

$P_a^b(f) = f(b) - f(a)$

$N_a^b(f) = 0$

$T_a^b(f) = f(b) - f(a)$

(2) $f\downarrow$ on $[a,b]$

$P_a^b(f)=0$

$N_a^b(f)=f(a)-f(b)$

$T_a^b(f)=f(a)-f(b)$

Summarized: f monotone on $[a,b]$

$\Rightarrow T_a^b(f)=|f(b)-f(a)|=P_a^b(f)+N_a^b(f)$

i.e., $f\in \mathrm{BV}[a,b]$

(3) $\mathrm{Max}\left\{N_a^b(f), P_a^b(f)\right\}\le T_a^b(f)\le P_a^b(f)+N_a^b(f)$.

(4) If $f\in \mathrm{BV}[a,b]$, then $P_a^b(f)+N_a^b(f)=T_a^b(f)$

$P_a^b(f)-N_a^b(f)=f(b)-f(a)$

(5) $a\le c\le b\Rightarrow T_a^b(f)=T_a^c(f)+T_c^b(f)\Rightarrow T_a^x(f)\uparrow$

Similarly for P & N

(6) $T_a^b(f+g)\le T_a^b(f)+T_a^b(g)$

(7) $T_a^b(cf)=|c|T_a^b(f)$

Note: $\mathrm{BV}[a,b]$ is a vector space.

Note: $T_a^b(\cdot)$ on $\mathrm{BV}[a,b]$ is "almost" a norm.

Except: $T_a^b(f)=0\Leftrightarrow f=\text{const. on }[a,b]$

(8) $f\in \mathrm{BV}[a,b]\Leftrightarrow f=g-h$, where $g,h\uparrow$ on $[a,b]$.

If f conti, g, h can be conti.

Pf: "$\Rightarrow$": Let $g(x)=P_a^x(f)$, $h(x)=N_a^x(f)-f(a)\uparrow$ (by (5))

Then $f(x)=g(x)-h(x)$ by (4)

"$\Leftarrow$": By (1), (6) & (7).

(cf. Ex.2.8.3)

(9) $f_n\to f$ pointwise on $[a,b]\Rightarrow T_a^b(f)\le \underline{\lim} T_a^b(f_n)$, i.e., $f\mapsto T_a^b(f)$ lower semiconti.

(10) f abso. conti. $\underset{\not\Leftarrow}{\Rightarrow} f$ of bdd variation (Ex.2.8.4)

Ex. $f\uparrow$ on $[a,b]$ but not conti.

Note: f unif. conti. $\not\Rightarrow f$ bdd variation (Ex.2.8.5)

(11) f satisfies Lipschitz condition $\Rightarrow f$ abso. conti. $\Rightarrow f$ of bdd variation on $[a,b]$.

i.e., $|f(x)-g(x)|\le M|x-y|\ \forall x,y\in[a,b]\Rightarrow T_a^b(f)\le M(b-a)$.

Homework: 2.8.2, 2.8.4, 2.8.5