Class 2

Chap.1. Measure theory

X set

$\{E_n\}$ subsets of X

Def. $\overline{\lim}E_n = \{x \in X : x \text{ belongs to infinitely many } E_n\text{'s}\}$

$$= \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

$\underline{\lim}E_n = \{x \in X : x \text{ belongs to all but finitely many } E_n\text{'s}\}$

$$= \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

Note: $\underline{\lim}E_n \subseteq \overline{\lim}\, E_n$

Def. $\{E_n\}$ has a limit if $\underline{\lim}E_n = \overline{\lim}E_n$

In this case, let $\lim E_n$ = this set

Ex. $E_n = \begin{cases} A \text{ if } n \text{ even} \\ B \text{ if } n \text{ odd.} \end{cases}$

Then $\overline{\lim}E_n = A \cup B$, $\underline{\lim}E_n = A \cap B$.

$\therefore \lim E_n$ exists $\Leftrightarrow A=B$

Analogue: $a,b,a,b,\ldots$. Then limit exists $\Leftrightarrow a=b$

Properties:

(1) $\underline{\lim}E_n \subseteq \overline{\lim}E_n$

$$\begin{array}{ccc} \underline{\lim}E_n & \subseteq & \overline{\lim}E_n \\ \cup\!\!\mid & & \cap\!\!\mid \\ \bigcap_n E_n & & \bigcup_n E_n \end{array}$$

(2) $(\underline{\lim}E_n)^c = \overline{\lim}E_n^c$ (Ex.1.1.1)

$(\overline{\lim}E_n)^c = \underline{\lim}E_n^c$

(3) $E_n \subseteq E_{n+1}$ $\forall_n \Rightarrow \lim E_n = \bigcup_n E_n$. (analogue: $a_n \uparrow$ & bdd above $\Rightarrow \lim a_n$ exists &$= \sup a_n$.)

$E_n \supseteq E_{n+1}$ $\forall_n \Rightarrow \lim E_n = \bigcap_n E_n$.

(4) $E \subseteq X$

Def. $\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}$

$\uparrow$

characteristic func. of E

$\chi_E : X \to \mathbb{R}$

$\overline{\lim}\,\chi_{E_n} = \chi_{\overline{\lim}\,E_n}$ (Ex.2.1.7)

$\underline{\lim}\,\chi_{E_n} = \chi_{\underline{\lim}\,E_n}$

(5) $\lim E_n$ exists $\Leftrightarrow \lim \chi_{E_n}$ exists.

Def. $\wp(X)$={all subsets of X}: power set of X

Def. $R \subseteq \wp(X)$ ring if

(1) $\phi \in R$

(2) $A, B \in R \Rightarrow A \setminus B \in R$ (Def. A\B={$x \in X$ & $x \notin B$})

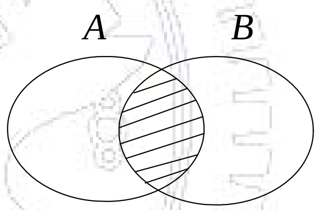
(3) $A, B \in R \Rightarrow A \cup B \in R$.

Properties of ring R:

(1) $A_1, .., A_n \in R \Rightarrow \bigcup_{i=1}^{n} A_i \in R$

(2) $A, B \in R \Rightarrow A \cap B \in R$

Pf: $A \cap B = A \setminus (A \setminus B) \in R$.



(3) $A_1, .., A_n \in R \Rightarrow \bigcap_{i=1}^{n} A_i \in R$

Def. R is an algebra if R ring & $X \in R$

(4) R ring

Define $A + B = A \Delta B$

$A \cdot B = A \cap B$.

Then $(R, \Delta, \cap)$ is an (algebaic) ring.

(cf. J.B. Wilker, Rings of sets are really rings, Amer. Math. Monthly,89 (1982), 211)

Def. R σ-ring if

(1) $\phi \in R$

(2) $A, B \in R \Rightarrow A \setminus B \in R$

(3) $A_1, A_2 ... \in R \Rightarrow \bigcup_{n=1}^{\infty} A_n \in R$

Def. R σ-algebra if R σ-ring & $\in$ R

Note: In prob. theory, elements in R are "events"; $X = \{\text{all outcomes}\}$

Ex. Toss a dice: $X = \{1,2,...,6\}$ $R = \wp(X)$

Note: σ-ring $\Rightarrow$ ring

σ-algebra $\Rightarrow$ algebra

Properties of σ-ring R:

(1) $A_1, A_2, ... \in R \Rightarrow \bigcap_{n=1}^{\infty} A_n \in R$. (Ex.1.1.3).

Let $A = A_1 \cup A_2 \cup .... \in R$

Pf. $\bigcap_n A_n = A \setminus \bigcup_n (A \setminus A_n) \in R$.

(2) $A_1, A_2, ... \in R \Rightarrow \overline{\lim} A_n, \underline{\lim} A_n \in$ R. (Ex.1.1.3).

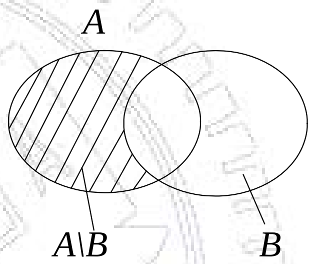
Thm 1. R is a ring $\Leftrightarrow \phi \in R$;

$A, B \in R \Rightarrow A \backslash B \in R$;

$A, B \in R, A \cap B = \phi \Rightarrow A \cup B \in R$.

Pf: "$\Leftarrow$"

$\because A \cup B = (A \backslash B) \cup B$



Thm 2. R is an algebra $\Leftrightarrow \phi \in R$;

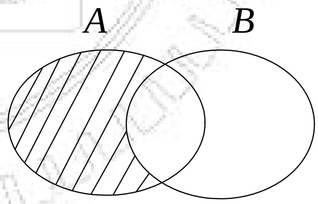
$A, B \in R \Rightarrow A \cup B \in R$;

$A \in R \Rightarrow A^c \in R$.

Pf: "$\Leftarrow$"

$\because A \setminus B = A \cap B^c = (A^c \cup B)^c \in R$

& $X = \phi^c \in R$



Thm 3. R is a σ-ring $\Leftrightarrow \phi \in R$;

$A, B \in R \Rightarrow A \backslash B \in R$;

$\{A_n\} \in R$, mutually disjoint $\Rightarrow \bigcup_n A_n \in R$

Thm 4. R is a σ-algebra $\Leftrightarrow \phi \in R$;

$A, B \in R \Rightarrow A \backslash B \in R$;

$\{A_n\} \subseteq R$, mutually disjoint $\Rightarrow \bigcup_n A_n \in R$

$X \in R$.

$D \subseteq \wp(X)$

Let R_0 be the intersection of all rings containing *D*.

Note1. intersection of rings is a ring

2. $\exists$ ring $\wp(X)$ which contains *D*.

$\Rightarrow$ R_0 is the smallest ring containing *D*.

Def. R_0 ring generated by *D*, denoted by *R*(*D*): .top-down;

buttom-up: perform $\cup,\cap,\backslash$ repeatelly on elements of *D*.

Similarly for σ-ring, algebra, σ-algebra.

Ex. $X = \mathbb{R}$

$D = \{[0,2],[1,3]\}$ $\qquad [0,2]\backslash[1,3] \quad [1,3]\backslash[0,2]$

|| $\qquad$ //

Then $R(D)=\{\phi,[0,2],[1,3],[0,1),(2,3],[1,2],[0,3],[0,1)\cup(2,3]\}$

|| $\quad$ || $\quad$ \\

σ-ring generated by $D = R(D)$ $\qquad [0,2]\cap[1,3] \quad [0,2]\cup[1,3]$

algebra generated by $D = R(D)\cup\{\mathbb{R}, (-\infty,0)\cup(2,\infty),...\}$

σ-algebra generated by *D*=algebra generated by $D\cup$....

Homework: Ex.1.1.8, 1.1.9