# Class 21

## Sec.2.9. DCT (dominated convergence thm)

Class 21

Sec.2.9. DCT (dominated convergence thm)

Question 1: $f_n \to f$, $f_n$ integrable $\not\Rightarrow f$ integrable;

Question 2: $f_n \to f$, $f_n, f$ integrable $\not\Rightarrow \int f_n - \int f$.

Thm. $\{f_n\}, g$ integrable.

$f_n \to f$ in meas. or a.e.

$|f_n| \le g$ a.e. $\forall n$

$\Rightarrow f$ integrable & $f_n \to f$ in mean. ($\Rightarrow \int f_n \to \int f$)

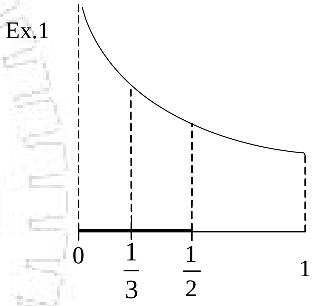
Note: a.e. or in meas. need extra condi. on $\{f_n\} \ni f$ integrable & $\int f_n \to \int f$

condi 1. $\{f_n\}$ Cauchy in mean.

condi 2. DCT: $|f_n| \le g$ a.e. $\forall n$, where $g$ integrable

condi 3. MCT: $0 \le f_n \uparrow f$ a.e.

Ex 1. $f_n(x) = \begin{cases} \frac{1}{x} & \text{if } \frac{1}{n} \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ for $x \in [0,1]$



Then $f_n$ integrable

But $f_n(\mathrm{x}) \to f(x)$ a.e. & in meas., but $f$ not integrable (Ex.2.7.6)

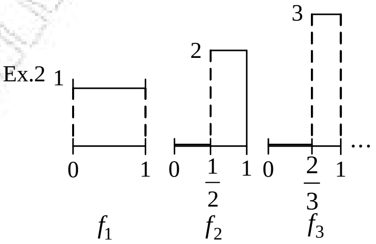
$$\| $$

$$\begin{cases} \frac{1}{x} & \text{if } 0 < x \le \frac{1}{n} \\ 0 & \text{if } x = 0 \end{cases}$$

Ex 2. (same as Ex.2.9.1) $f_n = n\chi[n-1/n, \pm]$ on $[0,1]$

Then $f_n \to 0$ a.e. & in measure on $[0,1]$

But $\int f_n = 1 \not\to \int 0 = 0$



Pf: (I) Assume $f_n \to f$ in meas.

Check: $\{f_n\}$ Cauchy in mean.

Then Thm 2.8.2 $\Rightarrow f$ integrable & $f_n \to f$ in mean.

Let $E = \bigcup_n \{x : f_n(x) \ne 0\}$

$$\therefore \int_X |f_n - f_m| = \int_{X\setminus E} |f_n - f_m| + \int_E |f_n - f_m| = \int_{F_k} |f_n - f_m| + \int_{E_k} |f_n - f_m|$$

$\|$ $\|$

$0$ $0$

$\wedge\backslash$

$\int_{F_k} |f_n| + \int_{F_k} |f_m|$

$\wedge\backslash$ $\wedge\backslash$

$\int_{F_k} g$ $\int_{F_k} g$

$\wedge$

$\varepsilon$ if $k$ large $\forall m, n$

Ex.2.6.2 $\Rightarrow E$ $\sigma$-finite

$\therefore E = \bigcup\limits_{k=1}^{\infty} E_k$, $E_k \uparrow$, in $\boldsymbol{a}$, $u(E_k) < \infty$.

Let $F_k = E \setminus E_k$

$\because F_k \downarrow \phi \Rightarrow \int_{F_k} g \to 0$ as $k \to \infty$

($\because$ finite meas.)

Now for $\int_{E_k} |f_n - f_m|$

Let $G_{mn} = \{x : |f_n(x) - f_m(x)| \geq \varepsilon_1\}$

$$\begin{aligned}\therefore \int_{E_k} |f_n - f_m| &= \int_{E_k \setminus G_{mn}} |f_n - f_m| + \int_{E_k \cap G_{mn}} |f_n - f_m| \\ &\leq \varepsilon_1 u(E_k \cap G_{mn}) + \int_{E_k \cap G_{mn}} |f_m| + \int_{E_k \cap G_{mn}} |f_n| \\ &\leq \varepsilon_1 u(E_k) + 2\int_{E_k \cap G_{mn}} g\end{aligned}$$

$\wedge$

$\varepsilon_2$ if $m, n$ large

Reason: $f_n \to f$ in meas.

$\Rightarrow \{f_n\}$ Cauchy in meas.

$\Rightarrow u(G_{mn}) \to 0$ as $m, n \to \infty$

$\Rightarrow \int_{G_{mn}} g$ small by abso. conti.

$\Rightarrow \int |f_n - f_m|$ small if $m, n$ large

i.e., $\{f_n\}$ Cauchy in mean.

(II) Assume $f_n \to f$ a.e. & $|f_n| \leq g$ a.e., g intergable

Check: $f_n \to f$ in measure . (Then by (I))

$\{x : |f_n - f| \geq \varepsilon\} \subseteq \bigcup\limits_{j \geq n} \{x : |f_j - f| \geq \varepsilon\} \equiv E_n.$

$\therefore u(\{x : |f_n - f| \geq \varepsilon\}) \leq u(E_n)$

Check: $u(E_n) \to 0$ as $n \to \infty$

Note: $E_n \downarrow \bigcap\limits_n E_n$ $\because$ $f_n \to f$ a.e.

$\Rightarrow u(\{x : f_n(x) \not\to f(x)\}) = 0$

$\cup /$

$\bigcap\limits_n E_n$

$\Rightarrow u(\bigcap\limits_n E_n) = 0$

Need to check: $u(E_n) < \infty$ (Then $u(E_n) \downarrow u(\bigcap_{n} E_n) = 0$)

$\therefore x \in E_n \Rightarrow$ for some $j \ge n,\ \varepsilon \le \left| f_j - f \right| \le \left| f_j \right| + \left| f \right| \le 2g \Rightarrow g \ge \frac{\varepsilon}{2}$ a.e.

($\because \left| f_n \right| \le$ g a.e. $\Rightarrow \left| f \right| \le$ g a.e.)

$\Rightarrow E_n \text{"} \subseteq \text{"} \left\{ x : g(x) \ge \frac{\varepsilon}{2} \right\}$

$\because g$ integrable $\Rightarrow$ RHS has finite measure.

$\Rightarrow u(E_n) < \infty$

Homework: Ex.2.9.2-2.9.4

↑

(integrable)