Class 22

Sec. 2.10 Applications of DCT. (easier to apply than def. of $\int f$).

Thm. f meas., g integrable.

$|f| \le g$ a.e. $\Rightarrow f$ integrable.

Note1: As comparison test for series or improper integral

2: If f simple, then Ex.2.7.2.

3: False for Riemann integrals:

Ex. $f(x) = \begin{cases} 1 \text{ if } x \text{ rational} \\ 0 \text{ if } x \text{ irrational} \end{cases}$ on [0,1]

Then $|f| \le 1$ on [0,1] But f not Riemann integrable

Pf: Check: $|f|$ integrable

$\exists$ simple $f_n \ni 0 \le f_n \uparrow |f|$ a.e.

$\therefore f_n \le |f| \le g$ a.e. & f_n simple

Ex. 2.7.2 $\Rightarrow f_n$ integrable

$\therefore$ DCT $\Rightarrow |f|$ integrable

Def. f meas. func.

f is essentially bdd if $\exists c > 0 \ni |f| \le c$ a.e.

Def. ess. sup$f = \inf \{c : |f| \le c \text{ a.e.}\}$

Cor 1. f integrable, g meas., essentially bdd $\Rightarrow fg$ integrable

Note: g may not be integrable

Pf: Say, $|g| \le c$ a.e.

$\therefore |fg| \le c|f|$ a.e.

$\uparrow$

integrable

$\Rightarrow |fg|$ integrable

$\Rightarrow fg$ integrable

Cor 2. $E \in \boldsymbol{a},\ u(E) < \infty$

f meas., esentially bdd on E

$\Rightarrow \int_E f$ exists.

Pf: $\because |f| \le c$ a.e. on E.

$\Rightarrow |\chi_E f| \le c\chi_E$ a.e.

$\uparrow$

integrable

$\Rightarrow \chi_E f$ integrable, i.e., $\int_E f$ exists.

(Note: f bdd on finite measure set $\Rightarrow f$ integrable

Much more general than Riemann integral

i.e., any proper integral conv.)

Monotone convergence thm: (MCT)

$0 \le f_n \uparrow f$ a.e., $\{f_n\}$ integrable

$\Rightarrow \int f_n \uparrow \int f$

Note: In general, f may not be integrable

Ex. $f_n(x) = \begin{cases} \frac{1}{x} & \text{on } [\frac{1}{n},1] \\ 0 & \text{on } [0,\frac{1}{n}) \end{cases}$ & $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$

Then $0 \le f_n \uparrow f$ a.e. & f_n integrable

But f not integrable, $\int_0^1 f = \infty$

Pf: (1) f integrable:

$\because 0 \le f_n \le f$ a.e.

DCT $\Rightarrow \int f_n \uparrow \int f$

(2) f not integrable:

Then $\int f = \infty$

Check: $\lim_n \int f_n = \infty$

Assume $\lim_n \int f_n < \infty$. ($\Rightarrow \{\int f_n\}$ Cauchy)

$\because \{f_n\}$ integrable, Cauchy in mean, $f_n \to f$ a.e.

$\uparrow$

(Reason: $\int |f_m - f_n| = \int f_m - \int f_n \to 0$ as $m,n \to \infty$)
(Assume $m \ge n$) ($\because \lim_n \int f_n$ exists)

$\Rightarrow f$ integrable $\quad \to\leftarrow$

Fatou's lemma:

$f_n \ge 0$, a.e. integrable $\forall n \Rightarrow \int \underline{\lim} f_n \le \underline{\lim} \int f_n$

Note: $f \mapsto \int f$ is lower semiconti.

($\because$ $f \mapsto \int f$ not conti. $\therefore$ we need DCT, MCT or Cauchy in mean)

Pf: Let $f = \underline{\lim} f_n = \sup_n \underbrace{\inf_{j \ge n} f_j}_{\substack{\parallel \\ g_n}}$

Then $0 \le g_n \uparrow f$ a.e. & g_n integrable ($\because 0 \le g_n \le f_n$ integrable)

$\therefore$ MCT $\Rightarrow \int g_n \uparrow \int f$

$\int g_n \le \int f_n$

$\Rightarrow \int f \le \underline{\lim} \int f_n$.

Note 1: In general, $\int f < \underline{\lim} \int f_n$ (Ex.2.10.14)

Ex. $f_n = X_{[n,n+1)}$ on $\mathbb{R}$, $f = 0$

Then $f_n \ge 0$, integrable, $f = \underline{\lim} f_n$ ($\because g_n = \inf_{j \ge n} f_j = 0$)

$\therefore \int f = 0 < \underline{\lim} \int f_n = 1$

Note 2: MCT $\underset{\Leftarrow}{\Rightarrow}$ Fatou (Ex.2.10.2)

Homework: Ex.2.10.2, Ex.2.10.3, Ex.2.10.4