Class 23- Class 24

Sec.2.11 (Proper) Riemann integral

f bdd function on $[a,b]$

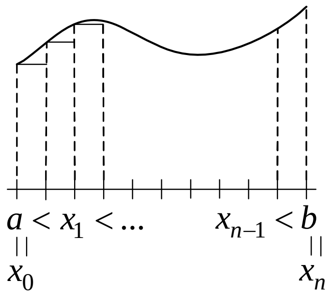
For any partition π: $a = x_0 < x_1 < \ldots < x_n = b$,

$|\pi| = \max\{x_i - x_{i-1} : 1 \le i \le n\}$

S_π upper Darboux sum

s_π lower Darboux sum

T_π Riemann sum



(1) Darboux integral: $\int_a^b f(x)dx \equiv \lim_{|\pi|\to 0} S_\pi = \lim_{|\pi|\to 0} s_\pi$

(2) Riemann integral: $\int_a^b f(x)dx \equiv \lim_{|\pi|\to 0} T_\pi$

Note: From advanced calculus, (1) & (2) the same.

(3) Lebesgue integral: $\int_{[a,b]} f(x)dx$

Thm 1. f bdd on $[a,b]$

Then f Riemann integrable iff f conti. a.e. on $[a,b]$

Ex. 1. f monotone on $[a,b]$

$\Rightarrow$ disconti. at most countable (Ex.2.11.2)

$\Rightarrow$ f Riemann integrable

Ex. 2. f of bdd variation on $[a,b] \Rightarrow f$ Riemann integrable

Thm 2. f bdd on $[a,b]$

Then f Riemann integrable $\Rightarrow$ f Lebesgue integrable & $\int_a^b f(x)dx = \int_{[a,b]} f(x)dx$

Note 1. not true on infinite interval

Ex. (Ex.2.11.3) $f(x) = \dfrac{\sin x}{x}$ on $(1,\infty)$

Then f Riemann integrable, but not Lebesgue integrable ($\because |f|$ not Riemann integrable)

Note 2. not true if f not bdd on $[a,b]$

Ex. $f(x) = \dfrac{\sin(\frac{1}{x})}{x}$ on $(0,1)$

(c.f. A.A. Kirillov & A.D. Gvishiani, Theorems and problems in functional analysis, Problem 191)

Then f Riemann integrable, but not Lebesgue integrable

Note 3. Riemann-Stieltjes & Lebesgue-Stieltjes (Ex. 2.11.9)

Pf of Thm 2:

(1) Check: f Lebesgue-measurable

(Then f bdd on $[a,b] \Rightarrow f$ Lebesgue integrable)

$$\begin{cases} \text{Check: } \forall \text{ open set } O,\ f^{-1}(O) \text{ Lebesgue measurable} \\ \text{Check: } \forall \text{ open interval } I,\ f^{-1}(I) \text{ Lebesgue measurable} \end{cases}$$

Let $E_1 = \{x \in (a,b) : f \text{ conti. at } x\}$

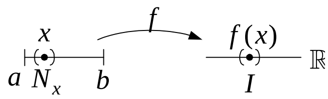
Let $E_2 = [a,b] \setminus E_1$. Then $m(E_2) = 0$

$$\because f^{-1}(I) = (f^{-1}(I) \cap E_1) \cup (f^{-1}(I) \cap E_2)$$

$$f^{-1}(I) \cap E_2 \subset E_2$$

$\Rightarrow f^{-1}(I) \cap E_2$ measurable

($\because$ Lebesgue measure complete)



Check: $f^{-1}(I) \cap E_1$ measurable

Let $x \in f^{-1}(I) \cap E_1$

Then $f(x) \in I$ and f conti. at x

$\Rightarrow \exists N_x$ nbd of $x \ni f(N_x) \subseteq I$

Let $N = \bigcup\limits_x N_x$

Then N open and $f^{-1}(I) \cap E_1 = N \cap E_1$: Lebesgue measurable

$\Rightarrow$ measurable

(2) Check: $\int_a^b f(x)dx = \int_{[a,b]} f(x)dx$

$\forall \pi: \ a = x_0 < x_1 < \ldots < x_{n\text{-}1} < x_n = b,$

Let $m_i = \inf\{f(x) : x \in (x_{i\text{-}1}, x_i)\}$

Let $f_\pi(x) = \sum\limits_{i=1}^{n} m_i \chi_{(x_{i\text{-}1},\, x_i)}$, simple, Lebesgue integrable

$\because f_\pi \le f$ a.e.

$\Rightarrow \int f_\pi \le \int f$

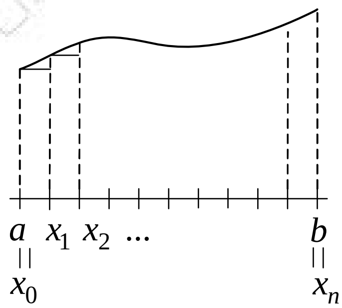
$\quad ||$

$\sum\limits_i m_i (x_i - x_{i-1}) = s_\pi$

$\Rightarrow \lim\limits_{|\pi| \to 0} s_\pi \le \int f$

$\quad ||$

$\int_a^b f(x)dx$



Similarly with upper sum $\Rightarrow \int_a^b f(x)dx \ge \int f$

$\Rightarrow \int_a^b f(x)dx = \int_{[a,b]} f(x)dx$

Homework: Ex.2.11.3, 2.11.4, 2.11.10

Sec. 2.12. Radon-Nikodym Thm.

(Motivation: f integrable on $(X,\boldsymbol{a},u)$ & $\mu(E)=\int_E f du$

$$\Rightarrow \frac{d\mu}{du}=f\text{: one half of fund. thm of calculus}$$

$(X,\boldsymbol{a})$ u, μ signed measures

Def. $\mu \ll u$ if $|u|(E)=0$ for some $E\in\boldsymbol{a} \Rightarrow \mu(E)=0$

(μ abso. conti. w.r.t. u).

Lma. The following are equiv.:

(a) $\mu \ll u$;

(b) $\mu^+ \ll u$ & $\mu^- \ll u$;

(c) $|\mu| \ll u$

Pf: Let $X=\mathrm{A}\cup\mathrm{B}$ be Hahn decomposition of μ

(a) $\Rightarrow$ (b):

Assume $|u|(E)=0 \Rightarrow |u|(E\cap A)=|u|(E\cap B)=0$

(a) $\Rightarrow \mu(\mathrm{E}\cap\mathrm{A})=\mu(\mathrm{E}\cap\mathrm{B})=0$

where $\mu(\mathrm{E}\cap\mathrm{A}) = \mu^+(E)$ and $\mu(\mathrm{E}\cap\mathrm{B}) = \mu^-(E)$

(b) $\Rightarrow$ (c):

Assume $|u|(E)=0$

Then $|\mu|(E)=\mu^+(E)+\mu^-(E)=0+0=0$

(c) $\Rightarrow$ (a):

Assume $|u|(E)=0$

$\therefore |\mu|(E)=0 \Rightarrow \mu^+(E)=\mu^-(E)=0 \Rightarrow \mu(E)=\mu^+(E)-\mu^-(E)=0$