Class 25

Thm. Assume $|u|(E) < \infty$ for $E \in a \Rightarrow |\mu|(E) < \infty$

Then $|\mu| \ll |u| \Leftrightarrow |\mu|$ abso. conti. w.r.t. $|u|$ (according to Def. 2.8.1 on p.52)

$\nwarrow\!\!\searrow$

Pf: $\forall \varepsilon > 0,\ \exists\ \delta > 0\ \ni\ \forall\ E \in \boldsymbol{a}\ \ni\ |u|(E) < \delta \Rightarrow |\mu|(E) < \varepsilon$

"$\Leftarrow$" Assume $|u|(E) = 0 \Rightarrow |\mu|(E) < \varepsilon\ \forall \varepsilon > 0$, i.e., $\mu \ll u$

"$\Rightarrow$" Assume contrary.

Then $\exists\ \varepsilon > 0,\ \forall \frac{1}{2^n} > 0\ \exists\ \{E_n\} \subseteq \boldsymbol{a}\ \ni\ |u|(E_n) < \frac{1}{2^n}$ & $|\mu|(E_n) \geq \varepsilon$

Let $E = \overline{\lim} E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$

Check: (1) $|u|(E) = 0$

(2) $|\mu|(E) \geq \varepsilon$ (Then, $|\mu| \not\ll u \Leftrightarrow \mu \not\ll u \quad \rightarrow\leftarrow$)

(1) $|u|(E) \leq |u|(\bigcup_{n=k}^{\infty} E_n) \leq \sum_{n=k}^{\infty} |u|(E_n) \leq \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}}\ \forall k$

$\Rightarrow |u|(E) = 0$

(2) $|\mu|(E) \geq \overline{\lim} |\mu|(E_n) \geq \varepsilon$

(Thm. 1.2.2)

(Reason: $|u|(\bigcup_{n=1}^{\infty} E_n) \leq \sum_n |u|(E_n) \leq \sum_n \frac{1}{2^n} = 1 < \infty$

$\Rightarrow |\mu|(\bigcup_{n=1}^{\infty} E_n) < \infty$

Note: If μ finite positive measure, then two def's are equiv.

Integration w.r.t. signed measure:

$(X, \boldsymbol{a})$, u signed measure

f meas. func. on X

Def: f integrable w.r.t. u if f integrable w.r.t. $|u|$

$\Updownarrow$

f integrable w.r.t. u^+ & u^-

Note: $\Downarrow$ Ex.2.12.1

$\Uparrow$ Ex.2.7.1

Def: $\int f du = \int f du^+ - \int f du^-$

Note: $(X, \boldsymbol{a})$ u signed measure

f integrable on X

Let $\mu(E) = \int_E f du$ for $E \in \boldsymbol{a}$

Then μ signed measure & μ abso. conti. w.r.t. u (as Def. 2.8.1 or Def. 2.12.1)

$\therefore$ Radon-Nikodym Thm says the converse

Thm. $(X, \boldsymbol{a})$

u, μ σ-finite signed measures

$\mu \ll u$ $(|u|(E) = 0 \Rightarrow \mu(E) = 0)$

Then (1) $\exists$ meas. f on X $\ni$ $\mu(E) = \int_E f du$ $\forall E \in \boldsymbol{a}$ $\ni |\mu|(E) < \infty$

(2) f is unique a.e. (w.r.t. u)   Note: f may not be integrable

Def: $\dfrac{d\mu}{du} = f$ a.e. $[u]$ (Radon-Nikodym derivative of μ w.r.t. u)

Note1: μ finite signed measure & $\mu \ll u$

Then f integrable w.r.t. u ($\Rightarrow \mu$ is indefinite integral of f)

Pf: $\because |\mu|(X) < \infty$

$\therefore \mu(X) = \int f du < \infty$

$\Rightarrow f$ integrable w.r.t. u

Note2: If u not σ-finite, R-N Thm not true.

Ex. $[0,1]$, $\boldsymbol{a} = \{\text{Lebesgue meas. sets}\}$

$u =$ counting measure

$\mu =$ Lebesgue measure

Then u not σ-finite ($\because$ $[0,1]$ uncountable), μ finite measure.

$\mu \ll u$

If f meas. $\ni$ $\mu(E) = \int_E f du$ $\forall E \in \boldsymbol{a}$

Let $E = \{x\}$

Then $\mu(\{x\}) = \int_{\{x\}} f du = f(x)$ $\forall x$

$\|$

0

$\Rightarrow f \equiv 0$

$\Rightarrow \mu \equiv 0$ $\rightarrow\leftarrow$

Note3: $u =$ Lebesgue measure on $[a,b]$,

$E = [a, x]$,

$F(x) \equiv \mu(E)$

$\Rightarrow \dfrac{d\mu}{du} = F'$

Note4: Change of variable (Ex.2.12.4); Chain rule (Ex.2.12.5); linearity (Ex.2.12.6)

Ex. Let $X=\{1,2,3,....\}$

$\boldsymbol{a}=2^X$

$u(\{k\})=1$ for $k\in X$

i.e., $u=$ counting measure

$\mu(\{k\})=\dfrac{1}{2^k}$ for $k\in X$

Then u σ-finite, μ finite, $\mu\ll u$. Find $\dfrac{d\mu}{du}$.

Solu. $\because \mu(E)=\int_E fdu$ for $E\in\boldsymbol{a}$. .

Let $E=\{k\}$

$\Rightarrow \dfrac{1}{2^k}=f(k)$

$\therefore \dfrac{d\mu}{du}(k)=\dfrac{1}{2^k}$ for $k\in X$

In general, $\dfrac{d\mu}{du}(k)=\dfrac{\mu(\{k\})}{u(\{k\})}$ ($\because \mu\ll u\Rightarrow$ well-defined a.e., $[u]$)

Similarly, $u\ll\mu$ & $\dfrac{du}{d\mu}=\dfrac{u(\{k\})}{\mu(\{k\})}$ $\forall k\in X$

Pf: Assume u, μ finite measures & $\mu\ll u$

(1) Existence:

Let $D=\left\{\hat{f}\geq 0,\text{ meas. }\&\int_E \hat{f}du\leq\mu(E)\ \forall E\in\boldsymbol{a}\right\}$

Note: $D\neq\phi$ (Reason: $\hat{f}\equiv 0\in D$)

Let $\alpha=\sup\left\{\int\hat{f}du:\hat{f}\in D\right\}$

Then $\exists\ \{f_n\}\subseteq D\ \ni\ \lim\limits_n\int f_n du=\alpha$

Let $g_n=\max(f_1,...f_n)$, $n=1,2,....$ & $f_0\equiv\sup\limits_n g_n$

Then $0\leq g_n\uparrow f_0$