Class 26

$\because f_n \in D \Rightarrow f_n$ integrable $\Rightarrow g_n =$ integrable (Ex.2.10.13)

$\therefore$ MCT $\Rightarrow \int g_n du \to \int f_0 du \Rightarrow \int f_0 du \ge \alpha$......(i)

$\vee /$

$\int f_n \to \alpha$

Check: $g_n \in D$

Check: $\int_E g_n du \le \mu(E) \quad \forall E \in \boldsymbol{a}$

Let $E_1 = \{x \in E : g_n(x) = f_1(x)\}$

$E_2 = \{x \in E : g_n(x) = f_2(x)\} \setminus E_1$

$\vdots$

$E_n = \{x \in E : g_n(x) = f_n(x)\} \setminus (E_1 \cup ... \cup E_{n\text{-}1})$

Then $E = \bigcup_{j=1}^{n} E_j$ & $\{E_j\}$ disjoint

$\therefore \text{LHS} = \sum_{j=1}^{n} \int_{E_j} g_n du = \sum_{j=1}^{n} \int_{E_j} f_j du \le \sum_{j=1}^{n} \mu(E_j) = \mu(E)$

Similarly, $\int_E g_n du \to \int_E f_0 du \ \forall E \in \boldsymbol{a}$

$\wedge \backslash$

$\mu(E)$

$\Rightarrow f_0 \in D$

$\Rightarrow \int f_0 du \le \alpha$......(ii)

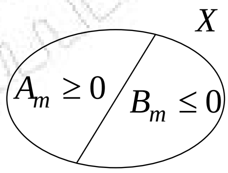
(i) & (ii) $\Rightarrow \int f_0 du = \alpha$

Let $\lambda(E) = \mu(E) - \int_E f_0 du$ for $E \in \boldsymbol{a}$. Then $\lambda \ge 0$.

Check: $\lambda \equiv 0$

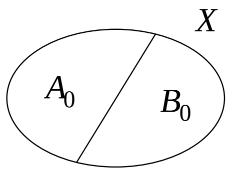
Consider $\lambda - \frac{1}{m} u$, $m = 1,2,...$, signed measure.

Let $X = A_m \cup B_m$ Hahn decomposition of $\lambda - \frac{1}{m} u$.



Let $A_0 = \bigcup_m A_m$, $B_0 = \bigcap_m B_m$

Then, $X = A_0 \cup B_0$ & $A_0 \cap B_0 = \phi$



$\because B_0 \subseteq B_m$ & $B_m \le 0 \ \forall m$

$\Rightarrow \lambda(B_0) - \frac{1}{m} u(B_0) \le 0 \ \forall m,$

$\therefore 0 \le \lambda(B_0) \le \frac{1}{m} u(B_0) \to 0$ as $m \to \infty$

$\Rightarrow \lambda(B_0) = 0$

Assume $\lambda \not\equiv 0 \Rightarrow \lambda(A_0) > 0$

$\because \lambda \ll u \Rightarrow u(A_0) > 0$

$(\because \lambda(E) = \mu(E) - \int_E f_0 du \ll u)$

$\Rightarrow u(A_m) > 0$ for some $m \geq 1$

Let $g = f_0 + \frac{1}{m}\chi_{A_m} \geq 0$

Check: $g \in D$

$\because \int_E g du = \int_E f_0 du + \frac{1}{m} u(E \cap A_m)$ $\qquad$ $A_m \geq 0$

$\wedge\backslash$ $\qquad$ $\cup|$

$\lambda(E \cap A_m)$ $\qquad$ $(\because (\lambda - \frac{1}{m}u)(E \cap A_m) \geq 0)$

$||$

$\mu(E \cap A_m) - \int_{E \cap A_m} f_0 du$

$\leq \mu(E \cap A_m) + \int_{E \backslash A_m} f_0 du$

$\leq \mu(E \cap A_m) + \mu(E \backslash A_m)$ $(\because \lambda \geq 0)$

$= \mu(E)$

$\Rightarrow \int g du \leq \alpha = \int f_0 du$

$||$

$\int f_0 du + \frac{1}{m} u(A_m)$

$\vee$

$\int f_0 du$ $\qquad \rightarrow\leftarrow$

$\therefore \lambda \equiv 0$

i.e., $\mu(E) = \int_E f du \ \forall_E \in \boldsymbol{a}$

(2) Uniqueness:

Assume $\mu(E) = \int_E f du = \int_E g du \ \forall E \in \boldsymbol{a}$

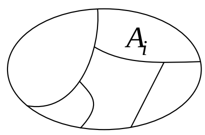
$\Rightarrow \int_E (f - g) du = 0 \ \forall_E \in \boldsymbol{a}$

Thm.2.7.6 $\Rightarrow f = g$ a.e. $[u]$

In general, assume u, μ σ-finite signed measures

i.e. $|u|, |\mu|$ σ-finite measures

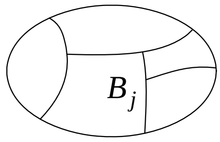
$\Rightarrow X = \bigcup_i A_i$, $|u|(A_i) < \infty$ & $\{A_i\}$ disjoint



$X = \bigcup_j B_j$, $|\mu|(B_j) < \infty$ & $\{B_j\}$ disjoint

$\Rightarrow X = \bigcup_{i,j}(A_i \cap B_j)$, $|u|(A_i \cap B_j) < \infty$, $|\mu|(A_i \cap B_j) < \infty$ & $\{A_i \cap B_j\}$ disjoint

$\Rightarrow \exists f_{ij}$ on $A_i \cap B_j$ $\ni$ $\mu(E) = \int_E f_{ij} du$ $\forall E \in \boldsymbol{a}$, $E \subseteq A_i \cap B_j$



Define $f = f_{ij}$ on each $A_i \cap B_j$

Homework: Ex.2.12.3~2.12.5

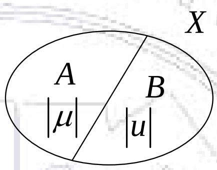
Sec. 2.13. Lebesgue decomposition (2 measures) (Hahn, Jordan decompositions) (only 1 measure)

$(X, \boldsymbol{a})$, u, μ signed measures

Def: $u \perp \mu$ if $\exists A, B \in \boldsymbol{a}$ $\ni$ $X = A \cup B$, $A \cap B = \phi$ & $|u|(A) = 0$, $|\mu|(B) = 0$

(u, μ mutually singular)

Ex: $X = [0,1]$, $\boldsymbol{a} = \{$Lebesgue meas. sets$\}$

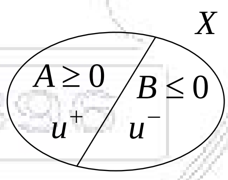


$u =$ Lebesgue measure

$$\mu(E) = \begin{cases} 1 \text{ if } \frac{1}{2} \in E \\ 0 \text{ otherwise} \end{cases} \quad \text{(point mass at } \tfrac{1}{2}\text{)}$$

Then $u \perp \mu$

Reason: $A = \left\{\frac{1}{2}\right\}$, $B = [0,1] \setminus \left\{\frac{1}{2}\right\}$



Properties:

(1) u signed measure, $u = u^+ - u^-$

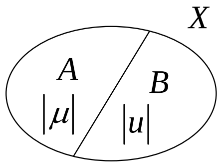
$\Rightarrow u^+ \perp u^-$, $u^+ \ll |u|$, $u^- \ll |u|$

$u^+(B) = u(A \cap B) = u(\phi) = 0$

$u^-(A) = u(B \cap A) = -u(\phi) = 0$

(2) $u \perp \mu_1$, $u \perp \mu_2 \Rightarrow u \perp (\alpha\mu_1 + \beta\mu_2)$ for $\alpha, \beta \in \mathbb{R}$ (Ex.2.13.2); $\mu_1 \ll u, \mu_2 \ll u \Rightarrow \alpha\mu_1 + \beta\mu_2 \ll u$

(3) $\mu \perp u$ & $\mu \ll u \Rightarrow \mu \equiv 0$



Pf: (i) $\mu \perp u \Rightarrow$

with $|\mu|(B)=0,\ |u|(A)=0$

(ii) $\mu \ll u \Rightarrow |\mu| \ll u$

$\because |u|(A)=0 \Rightarrow |\mu|(A)=0$

$\therefore \Rightarrow |\mu|(X)=0,$ i.e., $|\mu| \equiv 0$

$\Rightarrow \mu \equiv 0$

Thm. (Lebesgue decomposition)

u, μ σ-finite signed measures on $(X, \boldsymbol{\alpha})$

$\Rightarrow$ (1) $\exists$ σ-finite signed measures μ_0, μ_1 ∍

$\mu = \mu_0 + \mu_1,\ \mu_0 \perp u$ & $\mu_1 \ll u$

(2) μ_0, μ_1 are unique.