# Class 27

Note1: $\mu_0$, $\mu_1$ singular & abso. conti. parts of $\mu$ w.r.t. $u$

Pf: Assume $u, \mu$ finite positive measures

(1) Existence:

Let $\lambda = u + \mu$

i.e., $\lambda(E) = u(E) + \mu(E)\ \forall E \in \boldsymbol{a}$

$\therefore \mu \ll \lambda$

$\therefore$ R-N Thm $\Rightarrow \exists$ integrable $f \ni 0 \le \mu(E) = \int_E f\ d\lambda \le \lambda(E)\ \ \forall E \in \boldsymbol{a}$

$$\lambda(E) = \int_E 1 d\lambda$$

Ex: 2.7.3 $\Rightarrow 0 \le f \le 1$ a.e. $[\lambda]$

$\Rightarrow 0 \le f \le 1$ a.e. $[\mu]$

Let $A = \{x : f(x) = 1\}$

$B = X \setminus A$

Let $\mu_0(E) = \mu(E \cap A)$, $\mu_1(E) = \mu(E \cap B)$ finite measures

Note2: In prob. theory, $F(x) = P(X \le x)$: distribution function of random variable $X$

$u =$ Lebesgue measure on $\mathbb{R}$

$\mu =$ Lebesgue-Stieltjes measure corresponding to $F$ ($F \uparrow$)

$\therefore \mu = \mu_0 + \mu_1$, $\mu_1(E) = \int_E f(x)\ dx$

$\uparrow$ R-N derivative = density function of $X$ (for $f(x)$)

(i) $\because \mu = \mu_0 + \mu_1$ ($\because \mu(E) = \mu(E \cap A) + \mu(E \cap B) = \mu_0(E) + \mu_1(E)$)

(ii) $\because \mu_0(B) = \mu(B \cap A) = \mu_0(\phi) = 0$

Check: $u(A) = 0$

$\because \mu(A) = \int_A f\ d\lambda = \lambda(A) = u(A) + \mu(A)$ & $\mu(A) < \infty$

$\Rightarrow u(A) = 0$

Hence $\mu_0 \perp u$

(iii) $\mu_1 \ll u$:

Assume $u(E) = 0$

$\because \mu_1(E) = \mu(E \cap B) = \int_{E \cap B} f\ d\lambda = \int_{E \cap B} f du + \int_{E \cap B} f d\mu$

where $\mu(E \cap B) = \int_{E \cap B} 1 d\mu$ and $\int_{E \cap B} f du = 0$ ($\because u(E \cap B) \le u(E) = 0$)

$\Rightarrow \int_{E \cap B} (1 - f) d\mu = 0$

$\because 1 - f > 0$ a.e. on $E \cap B$ $[\mu]$

Thm. 2.7.5 $\Rightarrow \mu(E \cap B) = 0$

$\mu(E \cap B) = \mu_1(E)$

(2) Uniqueness:

Assume $\mu = \overline{\mu_0} + \overline{\mu_1},\ \ \overline{\mu_0} \perp u\ \ \&\ \ \overline{\mu_1} \ll u$

$= \mu_0 + \mu_1,\ \mu_0 \perp u\ \ \&\ \ \mu_0 \ll u$

$\Rightarrow \mu_0 - \overline{\mu_0} = \overline{\mu_1} - \mu_1 \perp u\ \ \&\ \ \ll u$ (by property (2))

$\Rightarrow \mu_0 - \overline{\mu_0} = \overline{\mu_1} - \mu_1 = 0$ (by property (3))

i.e., $\mu_0 = \overline{\mu_0}\ \ \&\ \ \overline{\mu_1} = \mu_1$

In general, $u$, $\mu$ $\sigma$-finite signed measures.

Homework: 2.13.4, 2.13.2