Class 28

Sec. 2.14 Fundamental Thm of Calculus on $\mathbb{R}$:

(I) f Lebesgue integrable on $[a,b]$

Let $g(x)=\int_a^x f(t)dt$

Then g' exists a.e. & $g'=f$ a.e.

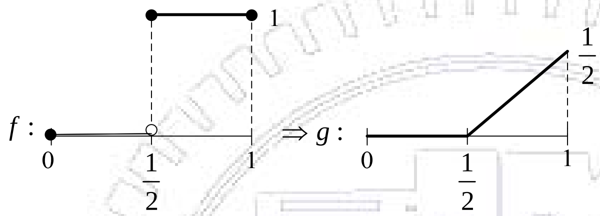
Note: f Riemann integrable on $[a,b]$

f conti. for some $x_0 \in (a,b)$

Then $g'(x_0)$ exists & $g'(x_0)=f(x_0)$

Ex.1. $f(x)=\begin{cases} 1 \text{ if } x \text{ rational} \\ 0 \text{ if } x \text{ irrational} \end{cases}$ not applicable in Riemann, applicable in Lebesgue.

Ex.2



$\therefore g'(\frac{1}{2})$ not exists ($\because f$ not conti. at $\frac{1}{2}$), but $g'=f$ a.e.

(Note: special case of (I))

Lma 1: f Lebesgue integrable on $[a,b]$

If $\int_a^x f(t)dt=0$ for all $x \in (a,b)$, then $f=0$ a.e.

$\Downarrow$

Pf: $\int_E f=0$ $\forall$ interval (open or closed) E

$\Downarrow$

$\int_E f=0$ $\forall$ open E (Reason: $E=\underset{i}{\cup} I_i$, I_i disjoint open intevals

$\Downarrow$ & $\int_E f=\int_{\underset{i}{\cup} I_i} f=\sum_i \int_{I_i} f=0$)

$\int_E f=0$ $\forall$ closed E (Reason: $\int_E f=\int_a^b f-\int_{[a,b]\setminus E} f=0-0=0$)

Assume Lebesgue integrable $f \neq 0$ a.e.

$\therefore \exists F \subseteq (a,b) \ni F$ Lebesgue measurable, $m(F)>0$ & $f(x)\neq 0$ for $x \in F$

$\Rightarrow \exists G$ closed in $[a,b]$, $G \subseteq F$, $m(G)>0$ & $f>0$ on G

($\because$ consider $\{x \in F : f(x)>0\}$ & $\{x \in F : f(x)<0\}$)

Thm.2.7.5 $\Rightarrow \int_G f>0$ $\rightarrow\leftarrow$

Lma 2: $f \uparrow$ on $[a,b]$.

Then f' exists a.e.

Note: f conti. a.e. (Ex.2.11.2)

Note: Two proofs:

(I) Use Vitali Lemma (measure theoretic proof)

(Ref. H. L. Royden, Real Analysis, Chap. 5., Sec. 1.)

Assume $\ell = a$ set of intervals in $\mathbb{R}$

$E \subseteq \mathbb{R}$

Def: ℓ covers E in the sense of Vitali if $\forall \varepsilon > 0$, $\underline{\forall x \in E,\ \exists I \in \ell \ \ni\ x \in I}$ & $m(I) < \varepsilon$

(usual covering)↗

Vitali Lemma

$m^*(E) < \infty$, ℓ covers E in the sense of Vitali.

Then $\forall \varepsilon > 0$, $\exists \{ I_1, ..., I_N \} \subseteq \ell$, disjoint $\ni$ $m^*(E \setminus \bigcup_{n=1}^{N} I_n) < \varepsilon$

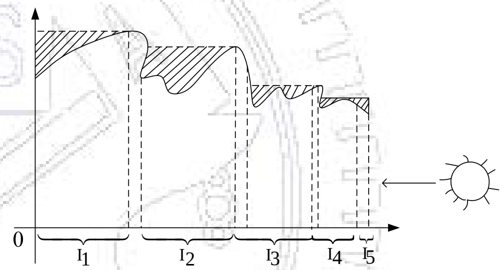
(II) Use F. Riesz's flowing water lemma (advanced calculus proof)

(rising sun lemma)

$g : I \to \mathbb{R}$ conti.

Then g has equal values at ends of I_i, $i = 2, 3, 4$

Intuition:



Summary: projection of mountain shades is open set with equal height

Lma:

Let g be conti. on interval I, except jumps.

Let $G(x) = \max\{g(x-),\ g(x),\ g(x+)\}$

Let $E = \{x \in I :\ \exists\, y \in I,\ y > x,\ g(y) > G(x)\}$

Then (1) E open,

(2) If $E = \bigcup_i (a_i, b_i)$, $\{(a_i, b_i)\}$ disjoint, then $g(a_i+) \le G(b_i)$.

(Ref. R.P. Boas, Jr., A primer of real functions, Sec. 22, pp.134-135)

Other uses of rising sum lemma:

(1) proof of Hardy-Littlewood maximal thm;

(2) Lebesgue's Thm on pts of density;

(3) Birkhoff ergodic thm.

Lma 3: f Lebesgue integrable on $[a,b]$

$g(x)=\int_a^x f(t)dt$

Then g is abso. conti. & of bdd variation & g' exists a.e.

Pf: By Ex.2.8.2 & Ex.2.8.4

Lma 4: f bdd & Lebesgue integrable on $[a,b]$

$g(x)=\int_a^x f(t)dt$

Then g' exists a.e. & $g'=f$ a.e. on $[a,b]$.

Pf: (1) Let $f=f^+-f^-$

$\therefore g(x)=\underbrace{\int_a^x f^+(t)dt}_{\uparrow}-\underbrace{\int_a^x f^-(t)dt}_{\uparrow}$

Lma 2 $\Rightarrow$ differ a.e.

$\Rightarrow g'$ exists a.e (without f bdd)

or Lma 3 $\Rightarrow g=g_1-g_2$, where g_1,g_2 $\uparrow$

$\therefore$ Lma 2 $\Rightarrow g'$ exists a.e

(2)Let $|f|\le$ M on $[a,b]$ (main idea:reduces to fundamental thm for Riemann integrals)

Extend f to $\mathbb{R}$ by defining $f(a)$, $f(b)$ beyond a,b

$$\because \left|\frac{g(x+h)-g(x)}{h}\right|=\left|\frac{1}{h}\left(\int_a^{x+h}f-\int_a^x f\right)\right|=\left|\frac{1}{h}\int_x^{x+h}f(t)dt\right|\le\frac{1}{|h|}\int_x^{x+h}|f|\le M\,\frac{(x+h)-x}{h}$$

$\uparrow$ assume $h>0$ $\qquad$ $\frac{(x+h)-x}{h}$ || M $\forall h$

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=g'(x) \text{ a.e. } \& \ \frac{1}{h}\left[g(x+h)-g(x)\right] \text{ conti.} \Rightarrow \text{integrable on [a,b]}$$

(Riemann $\Rightarrow$ Lebesgue)

g' meas. ($\because$ limit of conti. functions)

$$\text{DCT} \Rightarrow g' \text{ integrable } \& \int_a^x \frac{g(t+h)-g(t)}{h}dt \to \int_a^x g'(t)dt \ \ \forall x\in[a,b]$$

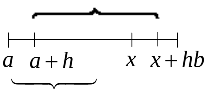
||

$$\frac{1}{h}\left(\int_{a+h}^{x+h}g(t)dt-\int_a^x g(t)dt\right)$$

||

$$\frac{1}{h}\left(\int_x^{x+h}-\int_a^{a+h}\right)\to g(x)-g(a)$$

(By fundamental thm. for Riemann integrals)



$$\Rightarrow \int_a^x g'(t)dt = g(x) - g(a) \quad \forall x \in [a,b]$$

$$\qquad\qquad \| \qquad\qquad \|$$

$$\int_a^x f(t)dt \qquad 0$$

$$\Rightarrow \int_a^x [g'(t) - f(t)]dt = 0 \quad \forall x \in [a,b]$$

Lma 1. $\Rightarrow g' = f$ a.e.

Lma 5. $\phi \uparrow$ on [a,b]

$\Rightarrow \phi'$ integrable, $\phi' \geq 0$ a.e. & $\int_a^b \phi' \leq \phi(b) - \phi(a)$

Note 1. "$<$" may occur.

Ex: $\varphi(x) = \begin{cases} 1 \text{ if } x = 1 \\ 0 \text{ if } 0 \leq x < 1 \end{cases}$

Then $\varphi \uparrow$ on [0,1] & $\varphi' = 0$ a.e.

$\therefore \int_0^1 \varphi' = 0 < \varphi(1) - \varphi(0) = 1$

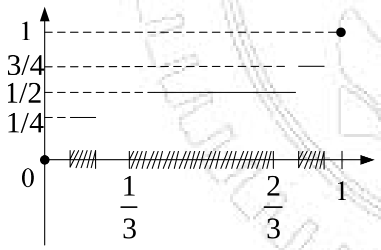
Note 2. "$<$" may occur even for φ conti. & $\uparrow$

Ex: (cf. K.L. Chung, A Course in probability theory, 1st edi., pp.12-13)

constructed from Cantor set: φ conti, $\uparrow$, $\varphi' = 0$ a.e. on [0,1], $\varphi(0) = 0$, $\varphi(1) = 1$

i.e., φ is singular function (cf. Ex.2.14.5)

$\therefore \int_0^1 \varphi' = 0 < \varphi(1) - \varphi(0) = 1$



Note 3. $\varphi \downarrow$ on $[a,b]$

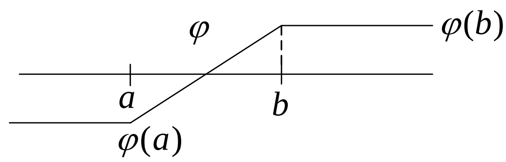
Then $-\varphi \uparrow$

$\Rightarrow \varphi'$ integrable $\varphi' \leq 0$ a.e. & $\int_a^b -\varphi' \leq -\varphi(b) + \varphi(a)$

$\Rightarrow \int_a^b \varphi' \geq \varphi(b) - \varphi(a)$

Pf: Define $\varphi(x) = \begin{cases} \varphi(b) \text{ if } x > b \\ \varphi(a) \text{ if } x < a \end{cases}$

Consider $\left\{ \dfrac{\varphi(x+h) - \varphi(x)}{h} \right\}_{h \neq 0}$



Then(1) meas.

Reason: $\varphi \uparrow \Rightarrow$ measurable (Ex.2.1.10)

(2) ≥ 0

Reason: $\varphi \uparrow$, consider $h > 0$ & < 0, separately.

(3) $\to \varphi'$ a.e. ($\Rightarrow \varphi'$ meas. & $\varphi' \geq 0$ a.e.)

Reason: Lma 2

$\therefore$ Fatou's Lma $\Rightarrow \int_a^b \varphi' \leq \underline{\lim}_{h \to 0^+} \int_a^b \frac{\varphi(x+h) - \varphi(x)}{h} dx$

($\because$ no DCT or MGT) $\qquad ||$

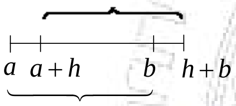
$$\underline{\lim} \frac{1}{h} \left(\int_b^{b+h} \varphi - \int_a^{a+h} \varphi \right)$$

$\wedge \backslash \qquad \forall h > 0$

$\varphi(b) - \varphi(a)$

$\because \varphi(x) \geq \varphi(a) \quad \forall x \in [a, a+h]$

$\Rightarrow \frac{1}{h} \int_a^{a+h} \varphi \geq \frac{1}{h} \varphi(a) \cdot (a+h-a) = \varphi(a)$



$\because \int_a^b \varphi' \leq \varphi(b) - \varphi(a) < \infty$

$\Rightarrow \varphi'$ integrable on $[a, b]$