Class 29
Fundamental Thm of Calculus:

(I) f Lebesgue integrable on [a,b]

$g(x)=\int_a^x f(t)dt$ for $x\in$[a,b]

Then $g'(x)$ exists a.e. & $g'=f$ a.e. on [a,b]

Pf: As in proof of Lma 4, g' exists a.e. & may assume $f\geq 0$

(Reason: $f=f^+-f^-$; $g(x)=\int_a^x(f^+-f^-)$, apply to f^+, f^-)

Let $f_n(x)=\begin{cases} f(x) & \text{if } f(x)\leq n \\ n & \text{otherwise}\end{cases}$

$\therefore f-f_n\geq 0$

$\Rightarrow \int_a^x(f-f_n)\uparrow$

Lma 2 $\Rightarrow \frac{d}{dx}\int_a^x(f-f_n)$ exists, ≥ 0 a.e.

i.e., $\frac{d}{dx}\int_a^x f\geq \frac{d}{dx}\int_a^x f_n \quad \forall n$

$\quad\quad || \quad\quad\quad ||$

$\quad\quad g' \quad\quad\quad f_n$ (by Lma 4, since f_n bdd)

Let $n\to\infty \Rightarrow g'\geq f$ a.e.

On the other hand, $\because f\geq 0\Rightarrow g\uparrow$

Lma 5 $\Rightarrow \int_a^b g'\leq g(b)-g(a)=\int_a^b f$

$g'\geq f$ a.e. $\Rightarrow \int_a^b g'\geq \int_a^b f$

$\Rightarrow \int_a^b(g'-f)=0$

$\because g'\geq f$ a.e. $\Rightarrow g'=f$ a.e.

Fund. Thm of Calculus (II):

Riemann: g' exists on $[a,b]$ & conti. on $[a,b]$

$\Rightarrow \int_a^x g'(t)dt=g(x)-g(a)$ for $x\in[a,b]$

Lebesgue: g abso. conti. on $[a,b]$

$\Leftrightarrow g'$ exists a.e., g' integrable on $[a,b]$ & $\int_a^x g'(t)dt=g(x)-g(a) \quad \forall x\in[a,b]$

Pf: "$\Leftarrow$" $\because g(x)=g(a)+\int_a^x g' \Rightarrow g$ abso. conti. (Ex.2.8.2)

"$\Rightarrow$" $\because g$ abso. conti.

Ex.2.8.4 $\Rightarrow g$ of bdd variation

Ex.2.8.3 $\Rightarrow g=g_1-g_2$, where $g_1, g_2\uparrow$

$\because g_1, g_2$ exists a.e. & integrable (Lma 5)

$\Rightarrow g'=g_1'-g_2'$ exists a.e. & integrable

Let $\varphi(x)=g(x)-\int_a^x g'(t)dt$ on $[a,b]$

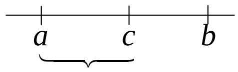
$\because \varphi'=g'-g'=0$ a.e.

$\Rightarrow \varphi=c$ on $[a,b]$ (Note: φ conti. & $\varphi'=0$ a.e. $\not\Rightarrow$ $\varphi=$ constant)

(Royden, p.109, Lma 13)

Lma. f abso. conti. on $[a,b]$, $f'=0$ a.e. $\Rightarrow$ $f=$ constant on $[a,b]$

Pf: Check: $f(c)=f(a)\ \forall c\in[a,b]$



Let $c\in[a,b]$

Let $E=\{x\in(a,c): f'(x)=0\}\Rightarrow m(E)=c-a$

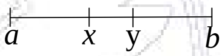
Need Vitali's lemma:

$$\left[\begin{array}{l} m^*(E)<\infty,\ \ell=\{\text{intervals}\}\ \ni\ \ell \text{ covers } E \text{ in the sense of Vitali} \\ \qquad\qquad \text{i.e., } \forall\varepsilon>0,\ x\in E,\ \exists I\in\ell\ \ni\ x\in I\ \&\ m(I)<\varepsilon \\ \text{Then } \forall\delta>0,\ \exists\{I_1,\dots,I_N\}\subseteq\ell \text{ disjoint } \ni\ m^*(E\setminus\bigcup_{n=1}^{N} I_n)<\delta \end{array}\right]$$

Let $\eta>0$ & $x\in E$

$\because f'(x)=0$

$\Rightarrow \exists\,[x,y]\subseteq[a,c]\ \ni\ |f(x)-f(y)|<\eta|x-y|$



Let $I=\{[x,y]\}$ covers E in the sense of Vitali

On the other hand, $\because f$ abso. conti.

$\therefore \forall\varepsilon>0,\ \exists\ \delta>0\ \ni\ \{[x_i,y_i]\}$ disj. $\ni\ \sum_{i=1}^{n}|y_i-x_i|<\delta\Rightarrow\sum_{i=1}^{n}|f(y_i)-f(x_i)|<\varepsilon$

For this δ, $\exists\ \{[x_1,y_1],\dots[x_n,y_n]\}\subseteq\ell$ disj. $\ni m^*(E\setminus\bigcup_{n=1}^{n}[x_i,y_i])<\delta$

May assume $\boxed{a\le x_1}<\boxed{y_1<x_2}<\dots<\boxed{y_n\le c}$, where $a = y_0$ and $c = x_{n+1}$

$\therefore \sum_{k=0}^{n}|x_{k+1}-y_k|<\delta$

$\Rightarrow \sum_{k=0}^{n}|f(x_{k+1})-f(y_k)|<\varepsilon$

$$\Rightarrow |f(c)-f(a)| = \left|\sum_{k=0}^{n}(f(x_{k+1})-f(y_k)) + \sum_{k=1}^{n}(f(y_k)-f(x_k))\right|$$
$$< \varepsilon + \sum_{k=1}^{n}\eta(y_k - x_k)$$
$$< \varepsilon + \eta(c-a)$$

Let $\varepsilon,\delta \to 0 \Rightarrow f(c)=f(a)$

$\because \int_a^b \varphi' \le \varphi(b)-\varphi(a) < \infty$

$\Rightarrow \varphi'$ integrable on $[a,b]$

$\therefore c = g(x) - \int_a^x g'(t)dt \ \ \forall x \in [a,b]$

Let $x=a \Rightarrow c = g(a)$

$\therefore \int_a^x g' = g(x)-g(a)$

Homework: Ex.2.14.5

Goal of Sec 2.15 & 2.16:

$f \ge 0$ (Tonelli) or $\iint |f(x,y)|dxdy < \infty$ (Fubini)

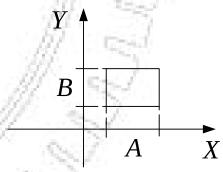
$\Rightarrow \iint f(x,y)dxdy = \int(\int f(x,y)dx)dy = \int(\int f(x,y)dy)dx$, i.e., double integral = interated integrals

Sec. 2. 15 Product of measures

$(X,\alpha)\ (Y,\beta),\ \ X\times Y,\ \ \alpha\times\beta$

Def. $\alpha\times\beta$ = the σ-algebra generated by $\{A\times B:\ A\in\alpha,\ B\in\beta\}$

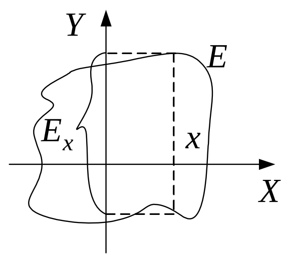
↑ Cartesian product of $\alpha \& \beta$ ↑ rectangle



Let E $\subseteq X\times Y$, $x\in X$, $y\in Y$

Def. $E_x = \{y:(x,y)\in E\}$ X-section of E

$E^y = \{x:(x,y)\in E\}$ Y-section of E



Lma 1 E $\in \alpha \times \beta$

$\Rightarrow E_x \in \beta$ & $E^y \in \alpha \quad \forall x \in X, y \in Y$

Pf: Let $D = \{F \in \alpha \times \beta :\ F_x \in \beta \ \forall x \in X\}$

Then D $\supseteq \{A \times B :\ A \in \alpha,\ B \in \beta\}$ & D σ-algebra

Reason: (1) $X \times Y \in D$
(2) $F \in D \Rightarrow F^c \in D \quad (\because (F_x)^c = (F^c)_x)$
(3) $A_n \in D \ \forall n \Rightarrow \bigcup_n A_n \in D \quad (\because (\bigcup_n A_n)_x = \bigcup_n (A_{n_x}))$

$\Rightarrow$ D $\supseteq \alpha \times \beta$

Lma 2 $Z \in X \times Y \in \alpha \times \beta,\ Z \subseteq X \times Y$

$\Gamma = \left\{ \bigcup_{i=1}^{n} (A_i \times B_i) :\ \{A_i \times B_i\} \text{ disjoint}, A_i \in \alpha, B_i \in \beta,\ A_i \times B_i \in Z \right\}$

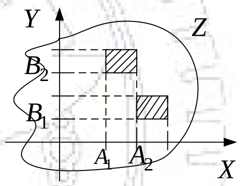
Then Γ ring

Pf: (1) $\phi \in \Gamma$

(2) Γ contains finitely disjoint union

(3) Check: $E, F \in \Gamma \Rightarrow E \setminus F \in \Gamma$

Say, $E = \bigcup_{i=1}^{n} E_i,\ F = \bigcup_{j=1}^{m} F_j$



$\therefore E \setminus F = (\bigcup_{i=1}^{n} E_i) \cap (\bigcap_{j=1}^{m} F_j{}^c)$

$= \bigcup_{i=1}^{n} \left[\bigcap_{j=1}^{m} E_i \cap F_j{}^c \right]$

$= \bigcup_{i=1}^{n} \left[\bigcap_{j=1}^{m} E_i \setminus F_j \right]$

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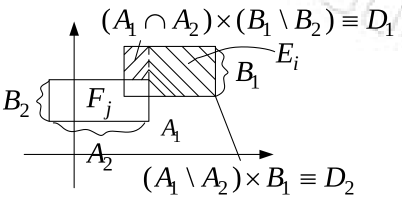
$\bigcap_{j=1}^{m} (D_{j_1} \cup D_{j_2}) \rightarrow$ disjoint rectangles



Say, $(D_{11} \cup D_{12}) \cap (D_{21} \cup D_{22})$

$= (D_{11} \cap D_{21}) \cup (D_{12} \cap D_{21}) \cup (D_{11} \cap D_{22}) \cup (D_{12} \cap D_{22})$

disjoint rectangles

$\Rightarrow \bigcap_{j} E_i \setminus F_j \in \Gamma$

$\cap|$

E_i

$\Rightarrow E \setminus F$ finite union of disjoint rectangles, $\in \Gamma$

$\Rightarrow \Gamma$ ring