Class 3

Sec.1.2. Measure

X set, $\boldsymbol{a}$ σ-algebra on X

Def. $u:\boldsymbol{a}\to[0,\infty]$ is a measure if

(1) $u(\phi)$=0;

(2) u is countably additive:

$$u(\overset{\infty}{\underset{n=1}{\cup}}E_n)=\sum_{n=1}^{\infty}u(E_n)\ \text{ for }\ E_n\in\boldsymbol{a}, E_i\cap E_j=\phi\ \text{ for }\ i\neq j$$

Def. additive: $u(E\cup F)=u(E)+u(F)$ for $E,F\in\boldsymbol{a}$, $E\cap F=\phi$

finitely additive: $u(E_1\cup\ldots\cup E_n)=u(E_1)+\ldots+u(E_n)$ for $E_1,\ldots,E_n\in\boldsymbol{a}$, $E_i\cap E_j=\phi$ for i ≠ j

subadditive: $u(E\cup F)\le u(E)+u(F)\quad\forall E,F\in\boldsymbol{a}$

finitely subadditive: $u(\bigcup_{i=1}^{n}E_i)\le\bigcup_{i=1}^{n}u(E_i)\ \forall E_1,\ldots,E_n\in\boldsymbol{a}$

countably subadditive: $u(\bigcup_{i=1}^{\infty}E_i)\le\bigcup_{i=1}^{\infty}u(E_i)\ \forall E_1,\ldots,E_n\in\boldsymbol{a}$

u finite measure if $u(X)<\infty$

u σ-finite measure if $\exists\ \{E_n\}\subseteq\boldsymbol{a}\ \ni\ u(E_n)<\infty\quad\forall n\ \&\ X=\underset{n}{\cup}E_n$

Properties for u measure on $\boldsymbol{a}$:

(1) $E_1,\ldots,E_n\in\boldsymbol{a},\ E_i\cap E_j\neq\phi$ for $i\neq j\Rightarrow u(\overset{n}{\underset{i=1}{\cup}}E_i)=\sum_{i=1}^{n}u(E_i)$ (finitely additive)

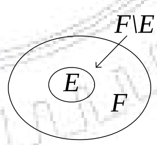
Pf: Let $E_{n+1}=E_{n+2}=\ldots=\phi$

(2) $E,F\in\boldsymbol{a},\ E\subseteq F\Rightarrow u(E)\le u(F)$.

Pf: $F=E\cup(F\setminus E)$, disj.

$(1)\Rightarrow u(F)=u(E)+u(F\setminus E)$.

$\Rightarrow u(E)\le u(F)$



(3) $E,F\in\boldsymbol{a},\ E\subseteq F,\ u(E)<\infty\Rightarrow u(F\setminus E)=u(F)-u(E)$.

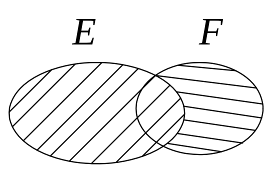
Note: If $u(E)=\infty$, then $u(F)=\infty\ \Rightarrow u(F)-u(E)$ meaningless.

(4) $E,F\in\boldsymbol{a}\Rightarrow u(E\cap F)+u(E\cup F)=u(E)+u(F)$ (Ex.1,2,3)

$\Rightarrow$ additives & ubaddiitive

Pf: $E\cup F=E\cup(F\setminus E)$, disj.

$\Rightarrow u(E\cup F)=u(E)+u(F\setminus E)$.



(1) $u(E\cap F)<\infty$: $u(F\setminus E)=u(F\setminus(E\cap F))$
$\quad\quad\quad\quad\quad\quad\quad\quad \| \leftarrow$ by(3)
$\quad\quad\quad\quad\quad\quad\quad\quad u(F)-u(E\cap F)$

(2) $u(E\cap F)=\infty$: $\therefore u(E\cup F)=\infty$ by (2)
$\quad\quad\quad\quad\quad\quad\quad\quad \therefore \text{LHS}=\infty=\text{RHS}$

(5) $E_n\in\boldsymbol{a}\Rightarrow u(\bigcap_n E_n)\le\sum_n u(E_n)$. (countably subadditive)

($\Rightarrow$ finitely subadditive, subadditive)

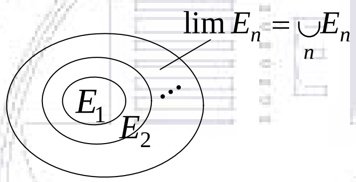
Pf: $F_1=E_1, F_2=E_2\setminus E_1, F_3=E_3\setminus(E_1\cup E_2),\ldots$

$\Rightarrow F_n\subseteq E_n\ \forall_n$ & $\bigcup_n F_n=\bigcup_n E_n$, $\{F_n\}$ mutually disjoint.

$\therefore u(\bigcup_n E_n)=u(\bigcup_n F_n)=\sum_n u(F_n)\le\sum_n u(E_n)$.

(6) $E_n\in\boldsymbol{a}$ & $E_n\uparrow\Rightarrow \lim u(E_n)=u(\lim E_n)$.

Pf:



$\because \lim E_n=E_1\cup(E_2\setminus E_1)\cup(E_3\setminus E_2)\cup\ldots$(disjoint union)

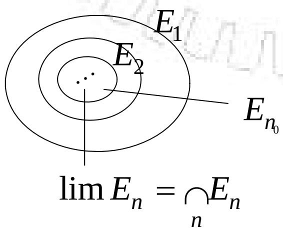
$\therefore u(\lim E_n)=u(E_1)+u(E_2\setminus E_1)+u(E_3\setminus E_2)+\ldots$

$=\lim\left[u(E_1)+u(E_2\setminus E_1)+\ldots+u(E_n\setminus E_{n\text{-}1})\right]$

$=\lim u(E_n)$.

(7) $E_n\in\boldsymbol{a}$ & $E_n\downarrow$, $u(E_{n_0})<\infty$ for some $n_0\Rightarrow \lim u(E_n)=u(\lim E_n)$.

Pf:



Consider E_{n_0} as universal set

$\because E_{n_0} \backslash E_n \uparrow, \in \boldsymbol{a}$ (for $n \ge n_0$)

$$(6) \Rightarrow \lim u(E_{n_0} \setminus E_n) = u(\bigcup_n (E_{n_0} \setminus E_n))$$

$$\lim(u(E_{n_0}) - u(E_n)) \quad\quad u(E_{n_0} \setminus \bigcap_n E_n)$$

$$u(E_{n_0}) - u(\bigcap_n E_n)$$

$$\Rightarrow \lim u(E_n) = u(\bigcap_n E_n)$$

Note: $u(E_{n_0})<\infty$ essential (cf. Ex.1.2.6)

(8) $E_n \in \boldsymbol{a} \Rightarrow u(\underline{\lim}\, E_n) \le \underline{\lim}\, u(E_n)$.

Pf: $u(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n) = \lim_k u(\bigcap_{n=k}^{\infty} E_n) \le \underline{\lim}_k\, u(E_k)$.

Note: This says $E \mapsto u(E)$ is lower semiconti.

(9) $E_n \in \boldsymbol{a}$, $u(\bigcup_n E_n)<\infty \Rightarrow u(\overline{\lim}\, E_n) \ge \overline{\lim}\, u(E_n)$.

Pf: $u(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n) = \lim_k u(\bigcup_{n=k}^{\infty} E_n) \ge \overline{\lim}_k\, u(E_k)$.

by (7) & $u(\bigcup_{n=1}^{\infty} E_n)<\infty$.

Note: $E \mapsto u(E)$ is upper semiconti. if u finite.

(10) $E_n \in \boldsymbol{a}$, $\lim E_n$ exists & $u(\bigcup_n E_n)<\infty \Rightarrow u(\lim E_n)=\lim u(E_n)$.

(i.e., u is conti. if u finite $\boldsymbol{a} \to [0,\infty]$)

Pf. $\overline{\lim}\, u(E_n) \le u(\overline{\lim}\, E_n) = u(\underline{\lim}\, E_n) \le \underline{\lim}\, u(E_n) \le \overline{\lim}\, u(E_n)$

$\Rightarrow u(\lim E_n)=\lim u(E_n)$.

Homework: Ex.1.2.5 & 1.2.6

Sec.1.3. Outer Measure

Motivation: Constructing Lebesgue measure;

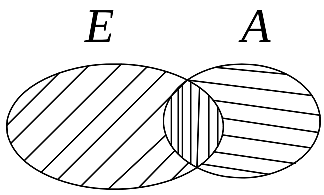
X set $\begin{cases} \text{convering sets by union of intervals \& taking inf.} \Rightarrow \text{gives outer measure.} \\ \text{Then measure.} \end{cases}$

Def. $u^*: \wp(X) \to [0,\infty]$ is outer mesaure if

(1) $u^*(\phi)=0$;

(2) u^* countably subadditive;

(3) $E, F \in \wp(X),\ E \subseteq F \Rightarrow u^*(E) \le u^*(F)$.



Outer measure $\to$ measure

Def. u^* outer measure on $\wp(X)$, $E \in \wp(X)$

E is u^*-measurable if $u^*(A) = u^*(A \cap E) + u^*(A \setminus E) \quad \forall A \subseteq X$

("$\le$" always true)