Class 30

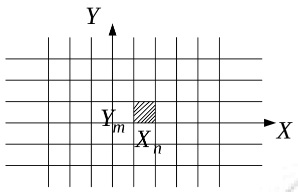
Lma 3 u, μ σ-finite measures on $(X, \alpha), (Y, \beta)$, resp.

Then $X \times Y = \bigcup_n X_n \times Y_m$, $\{X_n \times Y_m\}$ disjoint, $X_n \in \alpha$, $Y_m \in \beta$, $u(X_n) < \infty$, $\mu(Y_n) < \infty$

Pf: $\because X = \bigcup_n X_n$, $\{X_n\}$ disjoint, $X_n \in \alpha$, $u(X_n) < \infty$

$Y = \bigcup_m Y_m$, $\{Y_m\}$ disjoint, $Y_m \in \beta$, $\mu(Y_m) < \infty$

$\Rightarrow X \times Y = \bigcup_{n,.m} X_n \times Y_m$, $\{X_n \times Y_m\}$ disjoint



Def. $M \subseteq 2^X$ is monotone class if $\{E_n\} \subseteq M$, $E_n \uparrow$ or $\downarrow \Rightarrow \lim_n E_n \in M$

Lma 4. R monotone class & ring $\Rightarrow \sigma$-ring

Pf: Let $\{E_n\} \subseteq R$

Then $\bigcup_n E_n = E_1 \cup (E_1 \cup E_2) \cup (E_1 \cup E_2 \cup E_3) \cup .... \in R$

Def. $K \subseteq 2^X$

$M_0(K) = \cap L$, L monotone class & $L \supseteq K$

Note: (1) 2^X is a monotone class containing K

(2) The intersection of monotone classes is a monotone class

(3) If L monotone class & $L \supseteq K$, then $L \supseteq M_0(K)$

$\Rightarrow M_0(K)$ the smallest monotone class containing K

or the monotone class generated by K

Lma 5. $M \supseteq R$, $\Rightarrow M \supseteq S(R)$

↑ ↑ ↑

monotone ring σ-ring generated by R

Note: If $M = R$, then conclusion follows from Lma 4.

Pf of Lma 5.

Let $M_0 = M_0(R)$

Check: M_0 σ-ring (M_0 σ-ring $\supseteq R \Rightarrow M \supseteq M_0 \supseteq S(R)$)

Check: M_0 ring (Lma 4 $\Rightarrow M_0$ σ-ring)

Check: $E, F \in M_0 \Rightarrow E \setminus F,\ F \setminus E,\ E \cup F \in M_0$

$\forall F \subseteq M_0$, let $K_F = \{E \in M_0 : E \setminus F,\ F \setminus E,\ E \cup F \in M_0\}$

Check: $F \in M_0 \Rightarrow M_0 \subseteq K_F$

Properties:

(1) $E \in K_F \Leftrightarrow F \in K_E$

(2) K_F monotone class

Reason: $E_n \uparrow$ in $K_F \Leftrightarrow E_n \setminus F,\ F \setminus E_n,\ E_n \cup F \in M_0$

Check: $\bigcup_n E_n \in K_F \Leftrightarrow (\bigcup_n E_n) \setminus F,\ F \setminus (\bigcup_n E_n),\ (\bigcup_n E_n) \cup F \in M_0$

Check:(a) $(\bigcup_n E_n) \setminus F$

$\|$

$\bigcup_n (E_n \setminus F) \in M_0$

(b) $F \setminus (\bigcup_n E_n)$

$\|$

$\bigcap_n (F \setminus E_n) \in M_0$

(c) $(\bigcup_n E_n) \cup F = \bigcup_n (E_n \cup F) \in M_0$

Similarly for $E_n \downarrow$

Check: $F \in M_0 \Rightarrow R \subseteq K_F$ ($\because K_F$ monotone class)

Check: $F \in M_0, E \in R \Rightarrow E \in K_F$

$\Updownarrow$

$F \in K_E$

$\downarrow$

$\because E \in R \Rightarrow K_E \supseteq R$ $\Rightarrow K_E \supseteq M_0$ ($\because K_E$ monotone class)

Def. $\mathcal{D} \subseteq 2^X,\ A \subseteq X$,

$\mathcal{D} \cap A = \{D \cap A:\ D \in \mathcal{D}\}$

Lma 6. $S(\mathcal{D}) \cap A = S(\mathcal{D} \cap A)$

$\uparrow$ $\uparrow$

σ-ring σ-ring

Pf: "$\supseteq$":

$S(\mathcal{D}) \cap A \supseteq \mathcal{D} \cap A$;

$S(\mathcal{D}) \cap A$ is a σ-ring

$\Rightarrow S(\mathcal{D}) \cap A \supseteq S(\mathcal{D} \cap A)$

"$\subseteq$":

Let $K = \{B \cup (C \setminus A) : B \in S(\mathcal{D} \cap A),\ C \in S(\mathcal{D})\}$

$\uparrow$ $\cap|$

disjoint 2^A

Then(1) $K \supseteq \mathcal{D}$

Reason: $D \in \mathcal{D} \Rightarrow D = (D \cap A) \cup (D \setminus A)$

$\in S(\mathcal{D} \cap A)$ $\in S(\mathcal{D})$

(2)K σ-ring

(i) $\phi \cup (\phi \setminus A) \in K$

(ii)$[B_1 \cup (C_1 \setminus A)] \setminus [B_2 \cup (C_2 \setminus A)] = (B_1 \setminus B_2) \cup ((C_1 \setminus C_2) \setminus A) \in K$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in S(\mathcal{D} \cap A) \quad \in S(\mathcal{D})$

(iii)$\bigcup_n (B_n \cup (C_n \setminus A)) = (\bigcup_n B_n) \cup ((\bigcup_n C_n) \setminus A) \in K$

$\qquad\qquad\qquad\qquad\qquad\qquad \in S(\mathcal{D} \cap A) \quad \in S(\mathcal{D})$

$\Rightarrow K \supseteq S(\mathcal{D})$

$\Rightarrow K \cap A \supseteq S(\mathcal{D}) \cap A$

$\qquad ||$

$S(\mathcal{D} \cap A)$