Class 31

Thm. $(X,\ \alpha,\ u),(Y,\ \beta,\ \mu)$ σ-finite measure spaces.

$E\in\alpha\times\beta$

Let $f(x)=\mu(E_x)$ for $x\in X$: $X\to[0,\infty]$

$g(y)=u(E^y)$ for $y\in Y$: $Y\to[0,\infty]$

Then f,g meas. & $\int fdu=\int gd\mu$ $\leftarrow$ iterated integrals

Note: Special case of Tonelli's Thm: $h=\chi_E,\ f(x)=\int_Y\chi_E(x,\cdot)du,\ g(y)=\int_X\chi_E(\cdot,y)du$

Pf: Note: $E\in\alpha\times\beta\Rightarrow E_x\in\beta,\ E^y\in\alpha$ (Lma 1)

Let $\mathscr{D}=\{E\in\alpha\times\beta\text{: assertions true}\}$

Check: $\alpha\times\beta\subseteq\mathscr{D}$

(i) Let $A\in\alpha,\ B\in\beta$ with $u(A),\ \mu(B)<\infty$

Check: $\boxed{A\times B\in\mathscr{D}}$

$\therefore\ f(x)=\mu((A\times B)_x)=\begin{cases}\mu(B) & \text{if } x\in A=\mu(B)\chi_A\\ \mu(\phi)=0 & \text{if } x\notin A\end{cases}$

Similarly, $g(y)=u(A)\chi_B$

$\Rightarrow f,g$ meas.

& $\int fdu=\mu(B)u(A)=\int gd\mu$

(ii) Fix $E=A\times B\in\alpha\times\beta$ with $u(A),\ \mu(B)<\infty$

Let $D_E=\{D\in\mathscr{D}:D\subseteq E\}$

$\boxed{\text{Then } D_E \text{ monotone class}}$

Let $E_n\uparrow,\ E_n\subseteq E,\ E_n\in\mathscr{D}$

Check: $F=\bigcup_n E_n\in\mathscr{D}$

Pf: $\because E_n\uparrow F$

$\Rightarrow E_{nx}\uparrow F_x\Rightarrow\mu(E_{nx})\uparrow\mu(F_x)$

$$\begin{array}{cc}\| & \|\\ 0\le f_n(x) & f(x)\end{array}$$

$\because E_n\in\mathscr{D}\Rightarrow f_n$ meas. $\Rightarrow f$ meas.

MCT $\Rightarrow\int f_ndu\ \uparrow\ \int fdu$

Similarly for $\int g_nd\mu\ \uparrow\ \int gd\mu$

$\Rightarrow\int fdu=\int fd\mu$

i.e., $F\in\mathscr{D}$

Similarly for $E_n\downarrow$ (need: $u(A),\mu(B)<\infty$)

(iii) $E=A\times B\in\alpha\times\beta$ with $u(A),\ \mu(B)<\infty$

Check: $\boxed{(\alpha\times\beta)\cap E\in\mathscr{D}}$

Let $D_E=\{D\in\mathscr{D}:D\subseteq E\}\subseteq\mathscr{D}$.......(a)

(i) $\Rightarrow D_E\supseteq\{A\times B\in\alpha\times\beta,A\times B\subseteq E\}$ $\to$ rectangles true

(ii) $\Rightarrow D_E$ monotone class

$$F_E=\left\{\bigcup_{i=1}^{n}A_i\times B_i:\{A_i\times B_i\}\subseteq\alpha\times\beta\text{ disjoint \& }A_i\times B_i\subseteq E\right\}$$

(0) $\{E_n\}$ disj. $\subseteq \mathscr{D} \Rightarrow \bigcup_n E_n \in \mathscr{D}$

Pf: $\because f(x) = \mu(\bigcup_n E_n)_x) = \mu(\bigcup_n (E_{nx})) = \sum_n \mu(E_{nx}) = \sum_n f_n(x)$ meas.

Similarly for g

Moreover, $\sum_{i=1}^{n} f_i \uparrow f \underset{\Rightarrow}{MCT} \sum_i \int f_i \uparrow \int f$ & for $g \Rightarrow \int f = \int g$

$\Rightarrow D_E \supseteq F_E$ ($\because$ disjoint union) $\quad \rightarrow$ ring true

Lma 2. $\Rightarrow F_E$ ring

Lma 5. $\Rightarrow D_E \supseteq S(F_E)$.........................(b) $\qquad \rightarrow \sigma$-ring true

Let $K = \{A \times B: A \in \alpha, B \in \beta\}$

$\therefore K \cap E \subseteq F_E$

$\Rightarrow S(K \cap E) \subseteq S(F_E)$

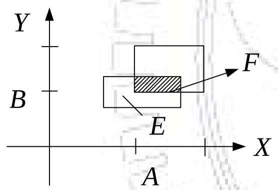
|| Lma 6

$S(K) \cap E$

|| Def of $\alpha \times \beta$

$(\alpha \times \beta) \cap E$.........................(c)

(a), (b) &(c) $\Rightarrow$ $(\alpha \times \beta) \cap E \subseteq \mathscr{D}$ $\qquad \rightarrow \alpha \times \beta$ restricted to E true



(iv) Check: $\alpha \times \beta \subseteq \mathscr{D}$

Let $F \in \alpha \times \beta$

Lma 3 $\Rightarrow F \subseteq \bigcup_{n=1}^{\infty} A_n \times B_n$: $\{A_n \times B_n\} \subseteq \alpha \times \beta$, disjoint & $u(A_n)$, $\mu(B_n) < \infty$

(iii) $\Rightarrow F \cap (A_n \times B_n) \in \mathscr{D}$

$\Rightarrow \bigcup_n [F \cap (A_n \times B_n)] \in \mathscr{D}$ ($\because$ disjoint union $\Rightarrow$ apply (0))

||

F

Thm. $(X, \alpha, u), (Y, \beta, \mu)$ σ-finite, Then

$X \times Y$, $\alpha \times \beta$

(1) $\lambda(E) = \int \mu(E_x) du = \int u(E^y) d\mu$ for $E \in \alpha \times \beta$: $\alpha \times \beta \rightarrow [0, \infty]$

is a σ-finite measure

(2) $\lambda(A \times B) = u(A) \cdot \mu(B)$ $\quad \forall A \times B \in \alpha \times \beta$

(3) λ is the unique measure satisfying (2)

Meaning: define $u \times \mu$ if integration vertically & horizontally give same answer

Notation: $\lambda = u \times \mu$

Pf: (1)(i)$\lambda\ (\phi) = \int \mu(\phi_x)du = \int \mu(\phi)du = 0$

(ii)$\lambda(\bigcup_i E_i) = \int \mu((\bigcup_i E_i)_x)du$ for disjoint $\{E_i\} \subseteq \alpha \times \beta$

$$= \int \mu(\bigcup_i E_{ix})du$$

$$= \int \sum_i \mu(E_{ix})du$$

$$= \sum_i \int \mu(E_{ix})du \quad \text{(MCT)}$$

$$= \sum_i \lambda(E_i)$$

$\Rightarrow \lambda$ measure

(2) $\lambda(A \times B) = \int \mu((A \times B)_x)du$

$$= \int \mu(B)\chi_A du$$

$$= \mu(B)u(A)$$

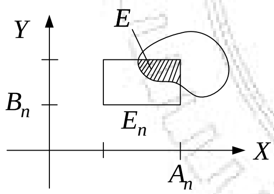
(1) $\because X = \bigcup_i A_i,\ u(A_i) < \infty,$ disjoint

$Y = \bigcup_j B_j,\ \mu(B_j) < \infty,$ disjoint

$\Rightarrow X \times Y = \bigcup_{i,j}(A_i \times B_j), \lambda(A_i \times B_j) = u(A_i)\cdot\mu(B_j) < \infty \Rightarrow \lambda\ \sigma$-finite

(3) Let $\bar{\lambda}$ be a measure $\ni \bar{\lambda}(A \times B) = u(A)\cdot\mu(B) \quad \forall A \times B \subseteq \alpha \times \beta$

Check:$\lambda = \bar{\lambda}$



same as proof of preceding thm:

$X \times Y = \bigcup_n E_n$, $\{E_n\} \subseteq \alpha \times \beta$, disjoint, $E_n = A_n \times B_n$, $u(A_n), \mu(B_n) < \infty$

Let $D_n = \{E \in (\alpha \times \beta) \cap E_n : \lambda(E) = \overline{\lambda}(E)\}$

Then D_n monotone class

> Reason: $G_m \in D_n \uparrow$ or $\downarrow$
>
> $\Rightarrow \lambda(\lim_m G_m) = \lim_m \lambda(G_m) = \lim_m \overline{\lambda}(G_m) = \overline{\lambda}(\lim_m G_m)$
>
> $\because \lambda(G_m), \overline{\lambda}(G_m) < \infty \ \forall m$

$K = \{A \times B \in \alpha \times \beta\}$

$\therefore D_n \supseteq K \cap E_n$ (Reason $:= \{$rectangles in $E_n\}$ & $\lambda = \overline{\lambda}$ on rectangles)

$\Rightarrow D_n \supseteq \{$finite disjoint unions from $K \cap E_n\} \equiv F_{E_n}$: ring

Lma 5 $\Rightarrow D_n \supseteq S(F_{E_n}) \supseteq S(K \cap E_n) = S(K) \cap E_n = (\alpha \times \beta) \cap E_n$

$\therefore \forall F \in \alpha \times \beta, F = \bigcup_n (E_n \cap F)$

$\Rightarrow \lambda(F) = \sum_n \lambda(E_n \cap F) = \sum_n \overline{\lambda}(E_n \cap F) = \overline{\lambda}(F)$

Note: $\int \mu(E_x) du \neq \int u(E^y) d\mu$ if u, μ not σ-finite

Ex: $X = Y = [0,1]$

$\alpha = \beta = \{$Lebesgue measurable subsets of $[0,1]\}$

$u =$ Lebesgue measure (finite)

$\mu =$ counting measure (not σ-finite)

Let $E = \{(x, y) \in X \times Y : x = y\}$

Then $E \in \alpha \times \beta$ ($\because E =$ intersection of small squares)

$\int \mu(E_x) du = \int \mu(\{x\}) du = \int 1 du = 1$

$\int u(E^y) d\mu = \int u(\{y\}) d\mu = \int 0 d\mu = 0$