Class 32

Note: $(\mathbb{R}^n,\ L^n,(dx)^n)$ σ-finite

$(\mathbb{R}^k,\ L^k,(dx)^k)$

$\Rightarrow (\mathbb{R}^n\times\mathbb{R}^k,\ L^n\times L^k,(dx)^n\times(dx)^k)$

$\| \int$

$(\mathbb{R}^{n+k},\ L^{n+k},(dx)^{n+k})$

Then (1) $\left\{\text{Borel sets in } \mathbb{R}^{n+k}\right\} \underset{\neq}{\subset} L^n\times L^k \underset{\neq}{\subset} L^{n+k}$

(Ex.2.15.4) (Ex.2.15.5)

(2) $(dx)^{n+k}|\left\{\text{Borel sets in } \mathbb{R}^{n+k}\right\}$, $(dx)^n\times(dx)^k$,$(dx)^{n+k}$ agree on common domains (Ex.2.15.6)

(3) $(dx)^{n+k}$ is the completion of $(dx)^n\times(dx)^k$. (i.e., u,μ complete $\not\Rightarrow u\times\mu$ complete)

(Ex.2.15.7)

Homework: Ex.2.15.2, 2.15.4, 2.15.5, 2.15.8

Sec. 2.16. Fubini's Thm

Assume $(X\times Y,\ \alpha\times\beta,\ u\times\mu)$ σ-finite

$h:\ X\times Y\to[-\infty,\infty],\ x\in X$

Def. $h_x(y)=h(x,y):\ Y\to[-\infty,\infty]$

x-section of h

Similarly, y-section of h: h^y

Lma. h meas. on $(X\times Y,\ \alpha\times\beta,\ u\times\mu)$

$\Rightarrow h_x,\ h^y$ meas. on Y, X, resp. $\forall x\in X$, $y\in Y$

Pf: Check: $h_x^{-1}(O)\in\beta$ $\forall$open $O\in\mathbb{R}'$

$\|$

$\left\{z\in Y: h_x(z)\in O\right\}$

$\|$

$\left\{z\in Y: h(x,z)\in O\right\}$

$\|$

$\left\{z\in Y: (x,z)\in h^{-1}(O)\right\}$

$\|$

$h^{-1}(O)_x\in\beta$ ($\because h^{-1}(O)\in\alpha\times\beta$)

(i.e., $h_x^{-1}(O)=h^{-1}(O)_x$)

Also, $h_x^{-1}(\{\infty\}),\ h_x^{-1}(\{-\infty\})\in\beta$

$\|$

$h^{-1}(\{\infty\})_x$

Tonelli's Thm: $(X,\alpha,u),(Y,\beta,\mu)$ σ-finite

$h: X\times Y \to [0,\infty]$ meas. on $X\times Y$

Then (1) $f(x)=\int h_x d\mu$ meas. on X;

(2) $g(y)=\int h^y du$ meas. on Y;

(3) $\int hd(u\times\mu)=\int(\underbrace{\int h^y du}_{g})d\mu=\int(\underbrace{\int h_x d\mu}_{f})du$ ($\le\infty$; may not be integrable)

(double integral) (iterated integrals)

Pf: (a) $h=\chi_E$ for some $E\in\alpha\times\beta$

Then $f(x)=\int(\chi_E)_x d\mu=\int\chi_{E_x}d\mu=\mu(E_x)$ meas. (by Thm. in preceding section)

$g(y)=u(E^y)$ meas.

& $\int gd\mu=\int fdu$ (by Thm)

$=(u\times\mu)(E)$ (by def)

$=\int hd(u\times\mu)$

(b) $h\ge 0$, simple meas.

Then, by linearity.

(c) $h\ge 0$, meas.

Let $0\le h_n\uparrow h$, h_n simple meas.

MCT $\Rightarrow \int h_n d(u\times\mu)\uparrow\int hd(u\times\mu)$

$\therefore 0\le h_{nx}\uparrow h_x$

MCT $\Rightarrow \int h_{nx}d\mu\uparrow\int h_x d\mu$

$$\begin{array}{cc} \text{|||} & \| \\ 0\le f_n(x) & f(x) \end{array}$$

$\Rightarrow f$ meas., i.e., (1) holds.

MCT $\Rightarrow \int f_n du\uparrow\int fdu$

Similarly for g: (2) holds & $\int g_n d\mu\uparrow\int gd\mu$

(b) $\Rightarrow$ (3) holds