Class 33

Fubini's Thm

$h: X\times Y \to \mathbb{R}$ integrable

Then (1) h_x integrable on Y for almost all x;

(2) h^y integrable on X for almost all y;

(3) $f(x)=\int h_x d\mu$ integrable on X, $g(y)=\int h^y du$ integrable on Y;

(4) $\int hd(u\times\mu)=\int\int hdud\mu=\int\int hdud\mu$ ($<\infty$)

Pf: Let $h=h^+-h^-$

$\therefore$ May assume $h\geq 0$, integrable

Tonelli's $\Rightarrow \int hd(u\times\mu)=\int gd\mu=\int fdu<\infty$ (i.e., (4))

$\Rightarrow f, g$ integrable (i.e., (3))

$\Rightarrow f, g$ finite a.e.

$\|$

$\int h_x d\mu$

i.e., h_x integrable for almost all x (i.e., (1))

Note: Difficult to obtain $u\times\mu \Rightarrow$ diffi. to determine the integrability of h

Combination of Tonelli & Fubini:

Cor.1. $h: X\times Y\to[-\infty,\infty]$ measurable

$\iint|h|dud\mu<\infty$ or $\iint|h|d\mu du<\infty$

Then h integrable on $X\times Y$ ($\Rightarrow$ Preceding Thm applicable)

Pf: Tonelli's for $|h|$ $\Rightarrow \int|h|d(u\times\mu)=\iint|h|dud\mu=\iint|h|d\mu du<\infty$

i.e., $|h|$ integrable

$\Rightarrow h$ integrable

Special cases:

Let $X=Y=\{1,2,3,\ldots\}$

$\alpha=\beta=2^X$

$u=\mu=$ counting mea.

Then $\alpha\times\beta=2^{X\times Y}$, $u\times\mu=$ counting measure ($\because \forall E\in\alpha\times\beta,\ E=\bigcup_{(i,j)\in E}\{(i,j)\}$)

(1) $h: X\times Y\to[0,\infty]$, i.e., $a_{mn}\geq 0$

$(m,n)\mapsto a_{mn}$

$\Rightarrow \sum_m\sum_n a_{mn}=\sum_n\sum_m a_{mn}$

(2) $\{a_{mn}\} \ni \sum_m \sum_n |a_{mn}| < \infty$ or $\sum_n \sum_m |a_{mn}| < \infty$

$\Rightarrow \sum_m \sum_n a_{mn} = \sum_n \sum_m a_{mn}$

Note 1. $h \geq 0$ in Tonelli's & h integrable in Fubini's are essential

Ex: $a_{ij} = \begin{matrix} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}$ Then, $\sum_i \sum_j a_{ij} = -2 \quad \sum_j \sum_i a_{ij} = 0$

(cf. Ex.2.16.3)

2. u, μ σ-finite essential (Example as given before)

Ex. Let $h = \chi_E$, where $E = \{(x, y) \in X \times Y : x = y\}$

Cor. 2. $E \in \alpha \times \beta$

Then $(u \times \mu)(E) = 0 \Leftrightarrow u(E^y) = 0$ for a.a. y

$\Leftrightarrow \mu(E_x) = 0$ for a.a. x

Pf: $(u \times \mu)(E) = 0$

$\|$

$\Leftrightarrow \int \mu(E_x) du = 0$

$\Leftrightarrow \mu(E_x) = 0$ a.a. x

Homework: Ex.2.16.1~3

Chap. 3. Metric spaces

Sec. 3.1 Topological spaces & metric spaces

Motivation:

Topological concepts: 20th century

Classical analysis: functions, limits

Modern analysis: spaces of functions

Motivation for topology:

Convergence→nbd→open sets→topology

X set

$K \subseteq 2^X$

Def: (X, K) topological space if

(1) $\phi, X \in K$;

(2) $A_1, \ldots, A_n \in K \Rightarrow \bigcap_{i=1}^{n} A_i \in K$;

(3) $A_\alpha \in K \Rightarrow \bigcup_\alpha A_\alpha \in K$

Def: open, closed sets

Def: (X, K) Hausdorff space if

$\forall x \neq y \in X$, $\exists A, B \in K \ni A \cap B = \phi \ \& \ x \in A, y \in B$.

Ex1.$(X, \{\phi, X\})$ not Hausdorff if $\# X \geq 2$ (indiscrete topology)

Ex2. $(\{0,1\}, \{\phi, \{0\}, \{0,1\}\})$ not Hausdorff (Sierpinski space)

Def: (X, K) normal space if

(1) Hausdorff;

(2) $\forall E \cap F = \phi$, E, F closed $\Rightarrow \exists A, B \in K \ni A \cap B = \phi \ \& \ E \subseteq A, F \subseteq B$

Def: neighborhood, closure, interior, subspace, compact, dense subset, nowhere dense, separable (countable dense subset), convergence $(\overline{Y} = X)$ $(\operatorname{int} \overline{Y} = \phi)$

Def: $Y \subseteq X$ sequentially compact if $\forall \{y_n\} \subseteq Y$, $\exists \{y_{n_k}\}$, $y \in Y \ni y_{n_k} \to y$

Note: In metric spaces, the following are equvalent: (cf. Thm. 3.5.4)

(1) Y sequentially compact;

(2) Y has Bolzano-Weierstrass property;

(3) Y compact;