Class 34

(X, ρ) metric space

Let $K = \{A \subseteq X : \forall x \in A, \exists N_x \subseteq A\}$

Then (X, K) topological space

Note: (X, K) normal space

Def: (X, ρ) is complete if Cauchy sequence converges

Note: (X, ρ) complete

Then $Y \subseteq X$ is complete $\Leftrightarrow Y$ closed

Ex. 1. $(\mathbb{R}^{\mathrm{n}}, \rho)$

$$\rho(x, y) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{\frac{1}{2}} \text{ if } x = (x_1, ..., x_n),\ y = (y_1, ..., y_n)$$

Then complete, separable metric space

(advanced calculus) ($\because$ pts with rational components)

Ex. 2. $\ell^\infty = \left\{(x_1, x_2, ...) : \sup_n |x_n| < \infty\right\}$

$$\rho(x, y) = \sup_n |x_n - y_n| \text{ if } x = (x_1, x_2, ..),\ y = (y_1, y_2, ...)$$

Then complete, metric space, not separable (large)

(Sec. 3.2) (Ex.3.1.7)

(Note. $\#\{(x_1, x_2, ...)\} : \sup_n |x_n| < \infty, x_n$ rational $= \aleph_1$)

Ex. 3. $\ell^p = \left\{(x_1, x_2, ...),\ \sum_n |x_n|^p < \infty\right\}$ $(1 \le p < \infty)$

$$\rho(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}} \text{ if } x = (x_1, x_2, ...),\ y = (y_1, y_2, ...)$$

Then complete, separable, metric space

(Sec. 3.2) (Ex.3.2.4) $\{(x_1, ..., x_n, 0, ...) : x_i \text{ rational}\}$

Ex. 4. $c = \left\{(x_1, x_2, ...) : \lim_{\mathrm{n}\to\infty} x_n \text{ exists}\right\} \subseteq \ell^\infty$

$\cup|$

$$c_0 = \left\{(x_1, x_2, ...) : \lim_{\mathrm{n}\to\infty} x_n = 0\right\}$$

Then complete, separable metric spaces (small) under $\|\ \|_\infty$

Ex. 5. $S=\{(x_1,x_2,...)\}$

$$\rho(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|x_n-y_n|}{1+|x_n-y_n|}.$$

Then $x^{(m)}\to x$ in $\rho \Leftrightarrow x^{(m)}\to x$ componentwise (Ex.3.1.4)

Then complete, separable metric space

(Ex.3.1.5) (Ex.3.1.6) (check)

Ex. 6. $C[a,b]=\{f:[a,b]\to\mathbb{R}$ or $\mathbb{C}$ conti.$\}$

$$\rho(f,g)=\sup_{t\in[a,b]}|f(t)-g(t)|$$

Then complete, separable metric space

(Ex.3.1.5) ($\because$ Weierstrass Thm $\Rightarrow$ polynomials are dense in $C[a,b]$

$\therefore$ Consider polynomials with rational coeffi.)

(X,ρ), $(X,\hat{\rho})$ metrics

Def: $\rho\sim\hat{\rho}$ if $\exists\ \alpha,\beta>0 \ni \alpha\hat{\rho}(x,y)\le\rho(x,y)\le\beta\hat{\rho}(x,y)\quad \forall x,y\in X$

Note: 1. "~" equivalence relation

2. $\rho\sim\hat{\rho} \underset{\not\Leftarrow}{\Rightarrow} \rho,\hat{\rho}$ induce the same topology

(Ex.3.1.2)

Pf: Let A be open w.r.t. ρ

$\therefore \forall x\in A,\ \exists N_x\subseteq A$

$$N_x=\{y\in X:\rho(x,y)<\delta\}\supseteq\left\{y\in X:\hat{\rho}(x,y)<\frac{\delta}{\beta}\right\}$$

$\Rightarrow A$ open w.r.t. $\hat{\rho}$

3. $\rho\sim\hat{\rho}$

Then (1) $x_n\to x$ in $\rho\Leftrightarrow x_n\to x$ in $\hat{\rho}$

(2) $\{x_n\}$ Cauchy in $\rho\Leftrightarrow\{x_n\}$ Cauchy in $\hat{\rho}$

(3) (X,ρ) complete $\Leftrightarrow (X,\hat{\rho})$ complete

Reason: by (1) & (2)

Note: In general, $x_n\to x$ in $\rho\Leftrightarrow x_n\to x$ in $\hat{\rho}$

$\not\Rightarrow \rho\sim\hat{\rho}$ (cf. Ex.3.1.2) i.e., equiv. metric $\underset{\not\Leftarrow}{\Rightarrow}$ same convergence

equiv. metric $\underset{\not\Leftarrow}{\Rightarrow}$ same topology

Ex. $\mathbb{R}^n$

$\rho_p(x,y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}$ metric for $p \geq 1$ (by Minkowski's $\leq$)

$\hat{\rho}(x,y) = \sup |x_i - y_i|$

Then ρ_p, $\hat{\rho}$ are equiv.

Reason: $\hat{\rho}(x,y) \leq \rho_p(x,y) \leq n^{\frac{1}{p}} \hat{\rho}(x,y) \quad \forall x,y \in \mathbb{R}^n$

$\therefore \rho_p \sim \hat{\rho} \quad \forall p \geq 1$

Ex. 7. $(X_1,\rho_1),\ldots,(X_m,\rho_m)$ metric spaces

$(X_1 \times \ldots \times X_m, \rho)$ product metric space

$\rho(x,y) = \sum_{i=1}^{m} \rho_i(x_i,y_i)$ if $x = (x_1,\ldots,x_m)$, $y = (y_1,\ldots,y_m)$ or $(\sum_i \rho_i(x_i,y_i)^p)^{\frac{1}{p}}$ $(p \geq 1)$

or $\max_i \rho_i(x_i,y_i)$

Then all are equiv.

Ex. 8. $(X_n,\ \rho_n)$, $n = 1,2,\ldots,$ metric spaces

$(X_1 \times X_2 \times \ldots, \rho)$ product metric space

$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(x_n,y_n)}{1+\rho_n(x_n,y_n)}$

Note 1. Ex. 5. is a special case.

Note 2. (X,ρ) metric $\Rightarrow$ (X,δ) metric, where $\delta(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}$ & $\delta(x,y) \leq 1 \quad \forall x,y \in X$

(cf. Ex. 3.1.1)

Note 3. ρ, δ induce same topology, but $\rho \not\sim \delta$ (cf. Ex. 3.1.2)

Homework: Sec. 3.1, Ex. 3.1.1, Ex. 3.1.2, Ex.3.1.7, Ex. 3.1.11