Class 35

Sec. 3.2. L^p spaces

$(X, \boldsymbol{a}, u)$ measure space, $p \ge 1$

Def: $L^p(X,u) = \left\{ f : X \to [-\infty,\infty],\ \text{meas.},\ \int |f|^p\, du < \infty \right\}$

$\|f\|_p = (\int |f|^p\, du)^{\frac{1}{p}}$ for $f \in L^p(X,u)$

Def: $L^\infty(X,\ u) = \{ f : X \to [-\infty,\infty],\ \text{meas.}, f \text{ essentially bdd on } X \}$

$\|f\|_\infty = \text{ess. sup.}|f|$ for $f \in L^\infty(X,\ u)$

Hölder's $\le$:

$1 \le p,\ q \le \infty,\ \dfrac{1}{p} + \dfrac{1}{q} = 1$

$f \in L^p$, $g \in L^q \Rightarrow f \cdot g \in L^1$ and $\|fg\|_1 \le \|f\|_p \cdot \|g\|_q$

Note: For $p = q = 2$, Cauchy-Schwarz $\le$: $\|(x, y)\| \le \|x\|_2 \cdot \|y\|_2$; if $0 < p < 1$, then $q < 0$.

Pf: (1) $p = 1,\ q = \infty$:

$\therefore \|fg\|_1 = \int |fg| \le \|g\|_\infty \cdot \int |f| = \|g\|_\infty \cdot \|f\|_1$

Note: "=" $\Leftrightarrow$ for a.a. x, either $f(x) = 0$ or $|g(x)| = \|g\|_\infty$

(2) $p = \infty,\ q = 1$: Similarly as (1)

(3) $1 < p < \infty,\ 1 < q < \infty$:

Morever, for $1 < p < \infty$

"=" iff $|f|^p = c|g|^q$ a.e. for some $c > 0$ or $g = 0$ a.e. or $f = 0$ a.e.

If $\|f\|_p = 0$, then $f = 0$ a.e. $\Rightarrow fg = 0$ a.e. $\therefore$ conclusion trivial

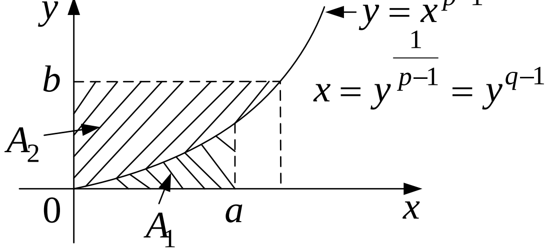
Similarly for $\|g\|_q = 0$

$\because$ Assume $\|f\|_p \cdot \|g\|_q > 0$

$\because ab \le \dfrac{a^p}{p} + \dfrac{b^q}{q}$ for $a, b \ge 0$ (Trivial for $p = q = 2$); Note: "=" iff $a^p = b^q$

(Young's $\le$; cf. Royden, p.123, Ex.8)

Pf. of Young's $\le$:



$\because A_1 = \int_0^a x^{p-1}dx = \dfrac{a^p}{p}$

$A_2 = \int_0^b y^{q-1}dy = \dfrac{b^q}{q}$

$\therefore A_1 + A_2 \ge a \cdot b$

Also, $"=" \Leftrightarrow a^{p-1} = b \Leftrightarrow a^p = ab = b^q$

Let $a = \dfrac{|f|}{\|f\|_p}$, $b = \dfrac{|g|}{\|g\|_q}$

$\Rightarrow \dfrac{|fg|}{\|f\|_p \|g\|_q} \le \dfrac{1}{p}\dfrac{|f|^p}{\|f\|_p^p} + \dfrac{1}{q}\dfrac{|g|^q}{\|g\|_q^q} \in L^1 \;\; \Rightarrow |fg| \in L^1$

$\therefore \dfrac{1}{\|f\|_p \|g\|_q}\int |fg| \le \dfrac{1}{p}\dfrac{1}{\|f\|_p^p}\int |f|^p + \dfrac{1}{q}\cdot\dfrac{1}{\|g\|_q^q}\int |g|^q = \dfrac{1}{p} + \dfrac{1}{q} = 1$

$\Rightarrow \|fg\|_1 \le \|f\|_p \cdot \|g\|_q$ & $"=" \Leftrightarrow \dfrac{|f|^p}{\|f\|_p^p} = \dfrac{|g|^q}{\|g\|_q^q}$ a.e. $\Leftrightarrow |f|^p = c \cdot |g|^q$ a.e. for some $c > 0$

or $g = 0$ a.e.

or $f = 0$ a.e.

Minkowski's $\le$

$1 \le p \le \infty$, $f, g \in L^p \Rightarrow f + g \in L^p$ & $\|f+g\|_p \le \|f\|_p + \|g\|_q$

Moreover, $"=" \Leftrightarrow f = 0$ a.e., $g = 0$ a.e., or $f = cg$ a.e. for some $c > 0$ (cf. Ex.3.2.7) (for $1 < p < \infty$)

Pf: (1) $p = 1$:

Then $\int |f+g| \le \int |f| + |g| = \int |f| + \int |g|$

i.e., $\|f+g\|_1 \le \|f\|_1 + \|g\|_1$

Note: $"=" \Leftrightarrow |f+g| = |f| + |g|$ a.e. $\Leftrightarrow fg \ge 0$ a.e.

(2) $p = \infty$:

ess. sup.$|f+g| \le$ ess. sup.$(|f|+|g|) \le$ ess. sup.$|f|$ + ess. sup.$|g|$

i.e., $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$

(3) $1 < p < \infty$:

$\because |f+g|^p \le (|f|+|g|)^p \le \begin{cases} (2|f|)^p & \text{if } |f| \ge |g| \\ (2|g|)^p & \text{if } |f| \le |g| \end{cases}$

$\Rightarrow |f+g|^p \le 2^p |f|^p + 2^p |g|^p$

$\Rightarrow f + g \in L^p$

$$\|f+g\|_p^p = \int |f+g|^p$$

$$\|$$

$$\int |f+g||f+g|^{p-1}$$

$\wedge\backslash$ "=" $\Leftrightarrow$ $fg \ge 0$ a.e. & for some $c_1, c_2 > 0$ $c_1|g|^p = |f+g|^{(p-1)q} = c_2|f|^p$ a.e.

$$\int |f|\cdot|f+g|^{p-1} + \int |g|\cdot|f+g|^{p-1}$$

$$\le \|f\|_p \cdot (\int |f+g|^{(p-1)q})^{\frac{1}{q}} + \|g\|_p (\int |f+g|^{(p-1)q})^{\frac{1}{q}} \quad \text{(Hölder's} \le)$$

$$\|$$

$$(\int |f+g|^p)^{\frac{1}{q}}$$

$$= \|f\|_p \cdot \|f+g\|_p^{\frac{p}{q}} + \|g\|_p \cdot \|f+g\|_p^{\frac{p}{q}}$$

$$= (\|f\|_p + \|g\|_p)\|f+g\|_p^{\frac{p}{q}}$$

$$\Rightarrow \|f+g\|_p^{p-\frac{p}{q}} \le \|f\|_p + \|g\|_p$$

$$\|$$

$$\|f+g\|_p$$

Note: False for $0 < p < 1$

Ex. $X = \{1,2\}$, $\boldsymbol{a} = \mathscr{P}(X)$, $u =$ counting meas.

$$\begin{cases} f(1)=1,\ f(2)=0 \\ g(1)=0,\ g(2)=1 \end{cases}$$

$$\|f+g\|_p = 2^{\frac{1}{p}} > \|f\|_p + \|g\|_p = 1+1 = 2$$

For $f, g \in L^p$ $(1 \le p \le \infty)$, define $\rho(f,g) = \|f-g\|_p = (\int |f-g|^p)^{\frac{1}{p}}$

(1) $\rho(f,g) \ge 0$

(2) $\rho(f,g) = 0 \Leftrightarrow f = g$ a.e. $\leftarrow \rho$ not metric

(3) $\rho(f,g) = \rho(g,f)$

(4) $\rho(f,g) \le \rho(f,h) + \rho(h,g)$

(Minkowski's $\le$ for $1 \le p \le \infty$)

In L^p, define $f \sim g$ if $f = g$ a.e.

Then "~" equivalence relation

Let $\overline{f}$ denote the equiv. class containing f

$\therefore L^p(X,u) = \left\{ \overline{f} \ : \ f \in L^p(X,u) \right\}$

For simplicity, $\overline{f}$ written as f

Def: $\overline{f}, \ \overline{g} \in L^p(X,u)$

$\rho(\overline{f},\overline{g}) = \| f - g \|_p$

Then (L^p, ρ) metric space if $1 \le p \le \infty$

Note: In general, (L^p, ρ) not metric space for $0 \le p < 1$ (Ex.3.2.5)