Real Analysis (II)

Textbook:

A. Friedman, Foundations of modern analysis (1970).

References:

(1) H. L. Royden, Real analysis, 3rd ed. (1989).
(2) B. Gelbaum, Problems in analysis (1982).
(3) J. B. Conway, A course in function analysis, 2nd ed. (1990).

Grades:

$$\left.\begin{array}{l}\text{(1) 平時成績(習題，點名)}\\ \text{(2) 期中考}\\ \text{(3) 期末考}\end{array}\right\}\text{各}\frac{1}{3}.$$

Contents:

(1) Metric Space Theory: (Chap. 3)

$$\begin{cases}\text{Advanced calculus: } R^n\\ \text{Real analysis: metric space}\end{cases}$$

Riesz-Fisher Thm (Sec. 3.2)
Arzela-Ascoli Thm (Sec. 3.6)
Stone-Weierstrass Theorem (Sec. 3.7)
fixed-ponit Theorem (Banach) (Sec. 3.8)

(2) Functional analysis principles (Chap. 4)
uniform boundedness principle (Sec. 4.5)
open mapping Theorem (Sec. 4.6)
Hahn-Banach Theorem (Sec. 4.8)

(3) Compact Operator Theory (Chap. 5)
Riesz-Schauder Theorem (Sec. 5.2)

(4) Hilbert Space Theory (Chap. 6)
Spectral theory of self-adjoint operators (Sec. 6.7)
(normal operators)

Class 36

$(X, \boldsymbol{a}, u)$ measure space, $1 \le p \le \infty$

$L^p(X,u) = \left\{ f : X \to \mathrm{R} \text{ measurable: } \int_X |f|^p \, du < \infty \right\}$ metric space under ρ

$\rho(f,g) = (\int_X |f-g|^p \, du)^{\frac{1}{p}}$ if $1 \le p \le \infty$; $\rho(f,g) = \text{ess. }\sup|f-g|$ if p = ∞

Thm: $(L^p(X,u), \rho)$ complete metric space for $1 \le p \le \infty$

(Riesz-Fisher, 1907)

Pf: (1) $1 \le p < \infty$:

Note: For $p = 1$, proved in Thm 2.8.3

Let $\{f_n\} \subseteq L^p$ be Cauchy in $\|\cdot\|_p$

Modify Lma 2.5.2 $\Rightarrow$ Cauchy in measure

$\forall \varepsilon > 0$, let $E_{n,m} = \{x \in X : |f_n(x) - f_m(x)| \ge \varepsilon\} = \left\{x \in X : |f_n(x) - f_m(x)|^p \ge \varepsilon^p \right\}$

$\because |f_n - f_m|^p$ integrable

$\Rightarrow u(E_{n,m}) < \infty$

Also, $|f_n - f_m|^p \ge \varepsilon^p \chi_{E_{n,m}} \Rightarrow \int |f_n - f_m|^p \ge \varepsilon^p u(E_{n,m})$

$\downarrow$

0

Thm 2.4.3 $\Rightarrow$ $\exists$ meas. $f \ni f_{n_k} \to f$ a.e.

(i) Check: $f \in L^p$

$\because |f_{n_k}|^p \to |f|^p$ a.e.

$\therefore$ Fatou's Lma $\Rightarrow \int |f|^p \le \underline{\lim}_k \int |f_{n_k}|^p \le \sup_n \int |f_n|^p < \infty$

$$\left(\begin{aligned} &\text{Reason: } \forall \varepsilon > 0, \exists N \ni \|f_n - f_N\|_p < \varepsilon \ \forall n \ge N \\ &\quad \therefore \|f_n\|_p \le \|f_n - f_N\|_p + \|f_N\|_p \ \ \forall n \ge N \\ &\qquad \le \varepsilon + \|f_N\|_p \end{aligned} \right)$$

$\Rightarrow f \in L^p$

(ii) Check: $f_n \to f$ in $\|\cdot\|_p$

i.e., $f_{n_k} \to f$ a.e. & $\{f_n\}$ Cauchy in $\|\cdot\|_p \Rightarrow f_n \to f$ in $\|\cdot\|_p$

$\because\ \|f_n - f\|_p \le \|f_n - f_{n_k}\|_p + \|f_{n_k} - f\|_p$

$\wedge$ $\qquad$ P

ε $\qquad$ $(\int |f_{n_k} - f|^p)^{\frac{1}{p}}$

for large n, n_k $\qquad$ $\wedge|$ (Fatou's Lma)

$(\underline{\lim}_{k} \int |f_{m_k} - f_{n_k}|^p)^{\frac{1}{p}}$

$\wedge$

ε

for large m_k, n_k

(2) $p = \infty$:

Let $\{f_n\} \subseteq L^\infty$ be Cauchy in $\|\cdot\|_\infty$

$\because |f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty$ a.e.

$\wedge\backslash$ for $m, n \ge N$

ε

$\Rightarrow \{f_n(x)\}$ Cauchy for a.a. $x \in X$

$\therefore\ f_n(x) \to f(x)$ a.e.

For fixed $n \ge N$,

let $m \to \infty$

$\Rightarrow |f_n(x) - f(x)| \le \varepsilon$ a.e. for $n \ge N$

$\Rightarrow \|f_n - f\|_\infty \le \varepsilon$

In parti, $|f(x)| \le |f_N(x)| + |f(x) - f_N(x)| \le \|f_N\|_\infty + \varepsilon$ a.e.

i.e., $f \in L^\infty$ & $f_n \to f$ in $\|\cdot\|_\infty$

History: Riesz: $L^2(0,1) \cong \ell^2$; Fisher: $L^2(0,1)$ complete

(March 18,1907) (March 5,1907)

Special cases

$X = \{1, 2, \ldots\}$

$\boldsymbol{a} = 2^X$

$u =$ counting measure

Then $f = g$ a.e. $\Leftrightarrow$ $f = g$

$\therefore L^p(X,u) = l^p = \left\{ (x_1, x_2, \ldots) : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$

$\therefore \|(x_1, x_2, \ldots)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$

Homework: Ex. 3.2.5, 3.2.6

Sec. 3.3 Completion of metric spaces

(X, ρ), (Y, δ) metric spaces.

Def. $f : X \to Y$ is isometric if $\delta(f(x), f(y)) = \rho(x, y)\ \forall x, y \in X$

Note: f isometric $\Rightarrow f$ 1-1 & conti.

Def. $f : X \to Y$ isomorphism if f isometric & onto. $\leftarrow$ metric same.

Def. $f : X \to Y$ homeomorphism if f is 1-1, onto, conti. & f^{-1} conti. $\leftarrow$ top same.

Note: (X, ρ), (Y, δ) isomorphic

$\Rightarrow X, Y$ homeomorphic
$\not\Leftarrow$

Ex. R with $\rho(x, y) = |x - y|$

$(-1,1)$ with $\delta(x, y) = |x - y|$

Then $f(x) = \dfrac{x}{1+|x|}$ homeo. from ℝ onto $(-1,1)$

But R complete & $(-1,1)$ not