Class 37

Thm: (X, ρ) metric space

Then $\exists$ metric space $(\hat{X}, \hat{\rho}) \ni$

(1) $(\hat{X}, \hat{\rho})$ complete;

(2) $\exists$ isometric $f: (X,\rho) \to (\hat{X}, \hat{\rho})$;

(3) $f(X)$ dense in $\hat{X}$.

Moreover, if $(\hat{X}, \hat{\rho})$ also satisfies (1), (2) & (3), then $(\hat{X}, \hat{\rho})$, $(\hat{X}, \hat{\rho})$ isomorphic.

Def. $(\hat{X}, \hat{\rho})$ completion of (X, ρ)

Pf: Let $\{x_n\}$, $\{y_n\}$ be Cauchy sequences in X

$\{x_n\} \sim \{y_n\}$ if $\rho(x_n, y_n) \to 0$ as $n \to \infty$

Then " $\sim$ " equivalence relation

Let $\tilde{x}$ denote the equivealence class containing $\{x_n\}$

Let $\hat{X} = \{\tilde{x}\}$

Let $\hat{\rho}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} \rho(x_n, y_n)$ if $\{x_n\} \in \tilde{x}$ & $\{y_n\} \in \tilde{y}$

Check: (i) limit exists;

(ii) $\hat{\rho}$ well-defined;

(iii) $\hat{\rho}$ metric;

(iv) $(X, \hat{\rho})$ complete;

Define: $f: X \to \hat{X}$ by $f(x)$ = the equiv. class containing the Cauchy seq. $\{x, x, x, \ldots\}$

(v) f isometric;

(vi) $f(X)$ dense in $\hat{X}$;

Define: $r: \hat{X} \to \hat{X}$ by $r(\hat{\hat{x}})$ = the equiv. class containing $\{x_n\}$,

where $\mathrm{f}(x_n) \to \hat{\hat{x}}$ in $\hat{\rho}$ ($\because f(X)$ dense in $\hat{X}$)

Check: (vii) r well-defined; (viii) r isomorphism

Ex.1. (X, ρ) complete metric space $\Rightarrow \hat{X} = X$.

Ex.2. $X = (0,1)$ with Euclidean metric

Then $\hat{X} = [0,1]$ with Euclidean metric

Ex.3. $X = \{\text{rational nos}\}$, Euclidean metric

Then $\hat{X} = \mathrm{R}$