Class 38

Sec.3.4. Complete metric spaces

Advanced Calculus: Cantor intersection theorem

$\phi \neq F_n$ compact, $\subseteq \mathbb{R}^n$, $F_n \downarrow \Rightarrow \bigcap_n F_n \neq \phi$

Note1. False if F_n not comapct

Ex.1. $F_n = (0, \frac{1}{n}) \subseteq \mathbb{R} \Rightarrow \bigcap_n F_n = \phi$

Ex.2. $F_n = [0, \frac{1}{n}] \subseteq \mathbb{R} \Rightarrow \bigcap_n F_n = \{0\}$

Note2. (X, ρ) metric space

$\phi \neq F_n \downarrow$ compact

$\Rightarrow \bigcap_n F_n \neq \phi$

Pf: Let $x_n \in F_n$

Then $\{x_n\} \subseteq F_1$ compact

F_1 sequentially compact

$\Rightarrow \exists\, x_{n_k} \ni x_{n_k} \to x$ in F_1

$\therefore \{x_{n_k}, x_{n_{k+1}}, \ldots\} \to x$

$\cap|$

F_{n_k} closed $\forall k$

$\Rightarrow x \in F_{n_k} \quad \forall k$

$\Rightarrow x \in F_n \quad \forall n$

Thm. (X, ρ) complete metric space

$\phi \neq F_n \subseteq X$ closed, $F_n \downarrow$, $d(F_n) \to 0 \Rightarrow \bigcap_n F_n = \{x\}$

Note: If only assume F_n bdd but $d(F_n) \not\to 0$, then $\bigcap_n F_n$ may be ϕ (Ex. 3.4.3 & 3.4.4)

Pf: Existence:

Let $x_n \in F_n$

$\because \rho(x_n, x_m) \leq d(F_n) \to 0$ if $m \geq n$ large

$\therefore \{x_n\}$ Cauchy sequence

$\Rightarrow \exists\, x \in X \ni x_n \to x$

$\in F_n$

$\Rightarrow x \in F_n \quad \forall n \quad (\because \{x_n, x_{n+1}, \ldots\} \to x)$

$\Rightarrow x \in \bigcap_n F_n$ $\qquad \cap|$

F_n

Uniqueness:

Let $x, y \in \bigcap_n F_n \Rightarrow x, y \in F_n \ \forall n$

Then $\rho(x,y) \le d(F_n) \to 0$ as $n \to \infty$

$\therefore \rho(x,y) = 0 \ \Rightarrow x = y$

Baire Theory:

Def: (X, ρ) metric space

$Y \subseteq X$ of first Baire category in X if $Y \subseteq \bigcup_n X_n$, where X_n nowhere dense in X

$(\text{Int}\, \overline{X_n} = \phi)$

$Y \subseteq X$ of 2nd Baire category if not of 1st category

Note: Measurement of smallness of sets:

(1) set-theoretical: small cardinal no.

(2) measure-theoretical: null set

(3) top-theoretical: 1st category

Ex1. A set with large cardinality but small measure:

Cantor set: cardinal number $= \aleph_1$, but Lebesgue measure = 0, nowhere dense & 1st category

Let $I_{n,k} = (r_n - \frac{1}{2^{n+k}},\ r_n + \frac{1}{2^{n+k}})$, where $\{r_n\}$ rational numbers in [0,1], $n, k \ge 1$

Then $I \equiv \bigcap_k \bigcup_n I_{n,k}$ is 2nd category, but Lebesgue measure = 0

(cf: B. Gelbaum: p.129, Prob. 236)

Ex2. $Y = \mathbb{R}$ of 2nd category in $\mathbb{R}$ (next thm)

$Y = \mathbb{R} \subseteq \mathbb{R}^2$

Then Y is nowhere $\Rightarrow Y$ 1st category in $\mathbb{R}^2$ (cf. Ex. 3.4.5)

(Baire,1899)

Thm. (X, ρ) complete metric space

$\Rightarrow X$ 2nd category in X

Pf: Assume $X = \bigcup_n X_n$, where X_n nowhere dense $\forall n$

Let $x_0 \in X$

Consider $B(x_0, 1) = \{x \in X : \rho(x, x_0) < 1\}$

(1) $\because \operatorname{Int} \overline{X}_1 = \phi$

$\Rightarrow B(x_0,1) \not\subseteq \overline{X}_1$

Let $x_1 \in B(x_0,1) \setminus \overline{X}_1 = B(x_0,1) \cap \overline{X}_1^c$ open

$\Rightarrow \exists\ \overline{B(x_1,r_1)} \subseteq B(x_0,1) \cap \overline{X}_1^c$ & $r_1 < \frac{1}{2}$

(2) $\because \operatorname{Int} \overline{X}_2 = \phi$

$\Rightarrow B(x_1,r_1) \not\subseteq \overline{X}_2$

Let $x_2 \in B(x_1,r_1) \setminus \overline{X}_2 = B(x_1,r_1) \cap \overline{X}_2^c$ open

$\Rightarrow \exists\ \overline{B(x_2,r_2)} \subseteq B(x_1,r_1) \cap \overline{X}_2^c$ & $r_2 < \frac{1}{3}$

$\vdots$

$\Rightarrow \exists B(x_n,r_n) \ni \phi \neq \overline{B(x_n,r_n)} \downarrow$ & $r_n \to 0$, $\overline{B(x_n,r_n)} \subseteq \overline{X}_n^c \ \forall n$

By preceding thm, $\exists\ x \in \bigcap_n \overline{B(x_n,r_n)} \subseteq \bigcap_n \overline{X}_n^c$

$\therefore x \notin \overline{X}_n \quad \forall n$

$\Rightarrow x \notin \bigcup_n \overline{X}_n \quad \to\leftarrow$

Ex: $Y = \{\text{irrational no's}\} \subseteq \mathbb{R}$

Then Y 2nd category in R

Reason: If Y first category, then $\mathbb{R} = \{\text{rational}\} \cup \{\text{irrational}\}$ of 1st category $\to\leftarrow$

Note: X 2nd category

$Y \subseteq X$ 1st category

$\Rightarrow X \setminus Y$ 2nd category