Class 39

Morel: Use topology to study analysis

Application:

X complete metric space

$X=\bigcup_n Y_n$, Y_n closed $\Rightarrow$ Int $Y_n\neq\phi$ for some n

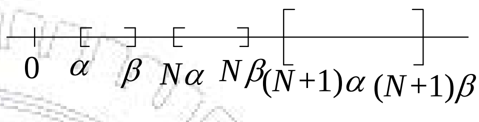
Ex: $f:[0,\infty)\to\mathbb{R}$ conti. & $\forall a\geq 0$, $f(na)\to 0$ as $n\to\infty$ (pointwise condi.)

Then $f(x)\to 0$ as $x\to\infty$

Pf: Fix $\varepsilon>0$

Let $Y_N=\{a\geq 0: \ |f(na)|\leq\varepsilon \ \forall n\geq N\} = \bigcap_{n=N}^{\infty} f(n\,\cdot)^{-1}([-\varepsilon,\varepsilon])$

Then $[0,\infty)=\bigcup_N Y_N$ & Y_N closed



$\because$ $[0,\infty)$ complete metric space $\Rightarrow$ 2nd category

$\Rightarrow \exists N \ni$ Int $Y_N\neq\phi$

$\therefore \exists\,[\alpha,\beta]\subseteq Y_N$ $(\alpha<\beta)$

$\therefore |f(x)|\leq\varepsilon \ \ \forall x\in[N\alpha,N\beta]\cup[(N+1)\alpha,(N+1)\beta]\cup\ldots$(local condi.)

Let k be $\ni$ $(N+k)\beta>(N+k+1)\alpha$, i.e., $k>\dfrac{1}{\beta-\alpha}(N\alpha+\alpha-N\beta)$

$\Rightarrow (N+k+1)\beta=(N+k)\beta+\beta>(N+k)\beta+\alpha>(N+k+1)\alpha+\alpha$

$=(N+k+2)\alpha$ etc.

Then $|f(x)|\leq\varepsilon \ \forall x\in[(N+k)\alpha,\infty)$, i.e., $\lim\limits_{x\to\infty} f(x)=0$ (global condi.)

Note1: In $(C[a,b],\|\cdot\|_\infty)$, the set of continuous nowhere differentiable functions is of 2nd category

i.e., Brownian motions are more typical

(A. Einstein, N. Wiener)

2. In $C^\infty[a,b]$ with some metric, the set of nowhere analytic functions is of 2nd category

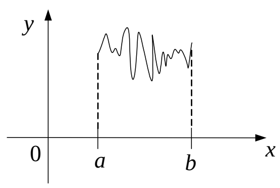
Def: $f\in C^\infty[a,b]$ is analytic at $x_0\in[a,b]$ if $\exists$ nbd N of x_0 $\ni$ $f(x)=\sum\limits_{n=0}^{\infty}a_n(x-x_n)^n$ on N

(cf: J. Dugundji, Topology, pp.300-302)

More precisely, the complements of the sets in (1) & (2) are of 1st category

Homework: Ex. 3.4.3, 3.4.4

Ex. $f(x)=\begin{cases} e^{-\frac{1}{x^2}} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$



Then $f\in C^\infty(-\infty,\infty)$. But f not analytic at 0

Ex: Let $f:[0,1]\to\mathbb{R}$ conti., $f_0=f$ and $f_n(x)=\int_0^x f_{n-1}(t)dt \ \ \forall n\geq 1$

$\forall x\in[0,1]$, $\exists\, n\geq 0$ $\ni$ $f_n(x)=0$. Then $f\equiv 0$ on $[0,1]$

Sec. 3.5 Compact metric spaces

Review:

(1) $Y \subseteq X$ (top. space) is compact if every open covering has a finite subcovering

(2) $Y \subseteq X$ (metric space) is sequentially compact if $\forall \{x_n\} \subseteq Y$, $\exists\, x_{n_k}$, $x \in Y \ni x_{n_k} \to x$

Thm (X, ρ) metric space

$K \subseteq X$ is sequentially compact

$\Rightarrow K$ is closed, bdd & separable

Pf: (1) K is closed:

Let $\{x_n\} \subseteq K$, $x_n \to x \in X$

Check: $x \in K$

$\because \exists x_{n_k} \to y \in K$ ($\because K$ sequentially compact)

But $x_{n_k} \to x$

$\Rightarrow x = y \in K$

(2) K is bdd: (i.e., $x, y \in K$, $\sup \rho(x,y) < \infty$)

Assume K not bdd

$\forall n$, $\exists\, x_n, y_n \in K \ni \rho(x_n, y_n) \geq n$

Fix $z \in X$

$\therefore \rho(x_n, z) + \rho(z, y_n) \geq \rho(x_n, y_n) \geq n$

$\Rightarrow \rho(x_n, z) \geq \dfrac{n}{2}$ or $\rho(z, y_n) \geq \dfrac{n}{2}$

$\therefore \exists \{x_n\} \subseteq K \ni \rho(x_n, z) \to \infty$ as $n \to \infty$

$\Rightarrow \exists\, x_{n_k} \in K \ni x_{n_k} \to x \in K$ ($\because K$ sequentially compact)

But $\rho(x_{n_k}, x) + \rho(x, z) \geq \rho(x_{n_k}, z)$

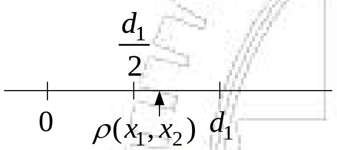
$$\underbrace{\rho(x_{n_k}, x)}_{\downarrow\; 0} \qquad \underbrace{\rho(x_{n_k}, z)}_{\downarrow\; \infty} \quad \to\leftarrow$$

Note: This also shows: $\forall z \in X, K \subseteq X$ sequentially compact $\Rightarrow \sup_{x \in K} \rho(z, x) < \infty$

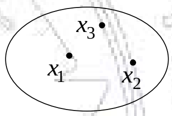
(3) K is separable

construction :
$$\begin{cases} \text{Let } x_1 \in K \\ \text{Let } d_1 = \sup\limits_{x\in K} \rho(x_1, x) < \infty \\ \text{Let } x_2 \in K \ni \rho(x_1, x_2) \geq \dfrac{d_1}{2} \\ \text{Let } d_2 = \sup\limits_{x\in K} \min\{\rho(x_1, x), \rho(x_2, x)\} < \infty \\ \text{Let } x_3 \in K \ni \min\{\rho(x_1, x_3), \rho(x_2, x_3)\} \geq \dfrac{d_2}{2} \\ \text{Let } d_3 = \sup\limits_{x\in K} \min\{\rho(x_1, x), \rho(x_2, x), \rho(x_3, x)\} < \infty \\ \text{Let } x_4 \in K \ni \min\{\rho(x_1, x_4), \rho(x_2, x_4), \rho(x_3, x_4)\} \geq \dfrac{d_3}{2} \end{cases}$$

$\Rightarrow \quad x_1, x_2, x_3, \ldots x_k, \ldots$



Motivation: Take $\{x_n\}$ separated far



$$\begin{cases} \text{Then } d_1 \geq d_2 \geq d_3 \geq \ldots \geq 0 \\ \text{If } \lim\limits_{n\to\infty} d_n = \delta > 0, \text{ then any } k < j \text{ satisfies } \rho(x_k, x_j) \geq \dfrac{1}{2} d_{j-1} \geq \dfrac{\delta}{2} > 0 \\ \qquad \Rightarrow \text{no subseq. of } \{x_n\} \text{ is Cauchy} \\ \qquad \rightarrow\leftarrow K \text{ sequentially compact} \\ \Rightarrow \lim\limits_{n\to\infty} d_n = 0 \end{cases}$$

Then $\{x_n\}$ dense in K

Reason: $\forall x_0 \in K,\ \forall \varepsilon > 0,\ \exists\ d_n < \varepsilon \Rightarrow B(x_0, d_n^+) \subseteq B(x_0, \varepsilon)$

If $x_1, \ldots, x_{n+1} \notin B(x_0, d_n^+)$,

then $\rho(x_0, x_i) \geq d_n^+ \Rightarrow d_{n+1} = \sup\limits_{x\in K} \min\limits_{i} \{\rho(x, x_i)\} \geq d_n^+ > d_n \quad \rightarrow\leftarrow$

$\Rightarrow \exists\ x_i \in B(x_0, d_n^+) \subseteq B(x_0, \varepsilon) \Rightarrow K$ separable

Thm. (X, ρ) metric space

$K \subseteq X$

Then K compact iff K sequentially compact

Cor. $K \subseteq X$ (metric space), compact $\Rightarrow$ bdd, closed, separable

Note: $K \subseteq X$ (complete metric space), bdd, closed $\not\Rightarrow$ compact

Ex. $X=\{1,2,....\}$

$$\rho(x,y)=\begin{cases}1 \text{ if } x\neq y\\ 0 \text{ if } x=y\end{cases}$$

Then (X,ρ) complete metric space

$K=X$ closed, bdd, but not compact, nor sequentially compact

($\because \left\{B(n,\frac{1}{2}):n\in X\right\}$ open covering, but no finite subcovering)

$=\{n\}$

($\{1,2,....\}$ has no conv. subseq.)