Class4

Thm. u^* outer measure

Let $\boldsymbol{a} = \left\{ u^* \text{-mesurable subsets} \right\}$

Then(i) $\boldsymbol{a}$ is σ-algebra;

(ii) $u^*|\boldsymbol{a}$ is measure.

Ex. X set. (Ex.1.3.2)

$$\text{Define } u^*(E) = \begin{cases} 0 & \text{if } \mathrm{E} = \phi \\ 1 & \text{if } E \neq \phi \end{cases} \quad \text{for } E \subseteq X.$$

Then u^* outer measure

$\boldsymbol{a} = \{\phi, X\}$

$u^*|\boldsymbol{a}$ is $\ni$ $u^*(\phi)=0$

$u^*(X)=1.$

Pf. of Thm.

(1) $\phi \in \boldsymbol{a}$

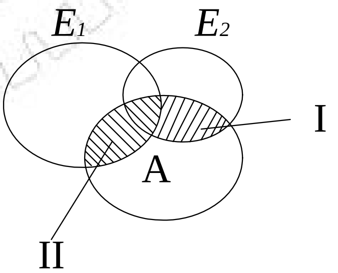
$$\begin{array}{lccl} \text{Check: } u^*(A) = & u^*(A\cap\phi) & + \, u^*(A\setminus\phi) & \forall A \subseteq X \\ & \parallel & \parallel & \\ & u^*(\phi) & u^*(A) & \\ & \parallel & & \\ & 0 & & \end{array}$$

(2) $E \in \boldsymbol{a} \Rightarrow E^c \in \boldsymbol{a}$

$$\begin{array}{lcc} \text{Check: } u^*(A) = & u^*(A\cap E^c) & + \, u^*(A\setminus E^c) \\ & \parallel & \parallel \\ & u^*(A\setminus E) & u^*(A\cap E). \end{array}$$

(3) $E_1, E_2 \in \boldsymbol{a} \Rightarrow E_1 \cup E_2 \in \boldsymbol{a}.$

$$\begin{array}{lcc} \text{Check: } u^*(A) \geq & u^*(A\cap(E_1\cup E_2)) & + \, u^*(A\setminus(E_1\cup E_2)) \\ & \parallel & \parallel \\ & u^*(\underset{\text{I}}{\underline{(A\setminus E_1)\cap E_2)}} \cup \underset{\text{II}}{\underline{(A\cap E_1)}}) & u^*((A\setminus E_1)\setminus E_2). \end{array}$$



$$\begin{aligned} \text{RHS} &\leq \underline{\underline{u^*((A\setminus E_1)\cap E_2)}} + u^*(A\cap E_1) + \underline{\underline{u^*((A\setminus E_1)\setminus E_2)}} \\ &= \underline{\underline{u^*(A\setminus E_1)}} + u^*(A\cap E_1) \qquad (\because E_2 \in \boldsymbol{a}) \\ &= u^*(A). \qquad (\because E_1 \in \boldsymbol{a}) \end{aligned}$$

From Sec.1.1, **a** is an algebra.

$\{E_k\} \subseteq \boldsymbol{a}$, mutually disjoint,

(4) $S_n = \bigcup_{k=1}^{n} E_k$

$$\Rightarrow u^*(A\cap S_n) = \sum_{k=1}^{n} u^*(A\cap E_k) \quad \forall A.$$

Pf: Trivial for $n=1$.

Assume true for $\le n$.

$\boldsymbol{a}$ σ-algebra.

&

countable

addition

$$u^*(A\cap S_{n+1}) = u^*((A\cap S_{n+1})\cap S_n) + u^*((A\cap S_{n+1})\setminus S_n) \ \ (\because S_n = \bigcup_{k=1}^{n} E_k \in \boldsymbol{a} \text{ by (3)})$$

$$= u^*(A\cap S_n) + u^*(A\cap \mathrm{E}_{n+1})$$

$$= \sum_{k=1}^{n} u^*(A\cap E_k) + u^*(A\cap \mathrm{E}_{n+1}) \quad \text{(by induction)}$$

$$= \sum_{k=1}^{n+1} u^*(A\cap E_k).$$

(5) Let $S = \bigcup_n E_n$. Then $u^*(A\cap S) = \sum_n u^*(A\cap E_n)$ $\forall A.$

Pf: "$\le$": $u^*(A\cap \mathrm{S}) = u^*(A\cap(\bigcup_n E_n)) = u^*(\bigcup_n (A\cap E_n)) \le \sum_{\mathrm{n}} u^*(A\cap \mathrm{E}_n).$

"$\ge$": $u^*(A\cap \mathrm{S}) \ge u^*(A\cap S_n) = \sum_{k=1}^{n} u^*(A\cap E_k)$ $\forall n.$ (by (4))

$$\downarrow$$

$$\sum_k u^*(A\cap E_k).$$

(6) $S \in \boldsymbol{a}$

Pf: Check: $u^*(A) \ge u^*(A\cap S) + u^*(A\setminus S).$ $\forall A.$

Pf: LHS=$u^*(A\cap S_n)$+$u^*(A\backslash S_n)$ $(\because S_n \in \boldsymbol{a}$ by (3))

$\| \leftarrow (4)$ $\rotatebox{-90}{$\geq$}$

$\sum_{k=1}^{n} u^*(A\cap E_k)$+ $u^*(A\setminus S).$

Let $n \to \infty$ $\downarrow$ $\|$

$\sum_n u^*(A\cap E_n)$+$u^*(A\backslash \mathrm{S}) \underset{(5)}{=} u^*(A\cap \mathrm{S})$+$u^*(A\backslash \mathrm{S}).$

(7) $\{E_n\} \subseteq \boldsymbol{a} \Rightarrow \bigcup_n E_n \in \boldsymbol{a}$

Pf: Let $F_1 = E_1,\ F_2 = E_2\setminus E_1,\ F_3 = E_3\setminus(E_1\cup E_2),\ldots$

Then $\{F_n\} \le \boldsymbol{a}$, disjoint

$\therefore \bigcup_n E_n = \bigcup_n F_n \in \boldsymbol{a}$ by (b)

$\Rightarrow \boldsymbol{a}$ σ-algebra.

$\because u^* | \boldsymbol{a}$ satisfies $u^*(\phi)$=0 & countable additivity (letting A=X in (5)) $\Rightarrow u^* | \boldsymbol{a}$ measure.

Homework: Ex.1.3.1, 1.3.3 &1.3.6

Sec.1.4. Constructing outer measure.

X set

$K \subseteq \wp(X)$

Def. K: sequential covering class if

(1) $\phi \in K$;

(2) $\forall \mathrm{A} \subseteq X,\ \exists \{E_n\} \subseteq K\ \ni\ \mathrm{A} \subseteq \bigcup_{n=1}^{\infty} E_n$.

Ex. $X = \mathbb{R}$

$K = \{\text{bdd open intervals}\} \cup \{\phi\}$

or $= \{\text{open intervals}\} \cup \{\phi\}$

or $= \{\text{bdd closed intervals}\} \cup \{\phi\}$

or $= \{\text{closed intervals}\} \cup \{\phi\}$.

Let $\lambda : K \to [0,\infty]$ & $\lambda(\phi)$=0

Ex. λ (interval)= its length

For $A \subseteq X$, let $u^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \lambda(E_n) :\ E_n \in K,\ A \subseteq \bigcup_n E_n \right\} : \wp(X) \to [0,\infty]$.

Note: In general, u^* may not be extension of λ (cf. p.14)

(Reason: u^* monotone, but λ may not be)

Thm. u^* is an outer measure.

Pf: (1) $u^*(\phi)$ =0

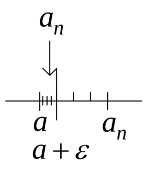
(2) $u^*(A) \le u^*(B)$ if $A \subseteq B$.

(3) Check: countable subadditivity

Let: $\{A_n\} \subseteq \wp(X)$.

Check: $u^*(\bigcup_n A_n) \le \sum_n u^*(A_n)$.

$$(\because\ a = \inf a_n \Leftrightarrow \begin{cases} a \le a_n \\ \forall \varepsilon > 0\ \ \exists a_n\ \ \ni\ a_n \le a+\varepsilon \end{cases})$$



For $\varepsilon > 0,\ \exists \{E_{nk}\} \subseteq K \to A_n \subseteq \bigcup_k E_{nk}$ & $\sum_k \lambda(E_{nk}) \le u^*(A_n) + \frac{\varepsilon}{2^n}$

$\because \bigcup_n A_n \subseteq \bigcup_{n,k} E_{nk}$

$$\Rightarrow u^*(\bigcup_n A_n) \le \sum_{n,k} \lambda(E_{n,k}) \le \sum_n (u^*(A_n) + \frac{\varepsilon}{2^n}) \le \sum_n u^*(A_n) + \varepsilon$$

Let $\varepsilon \to 0$