Class 40

Def. $K \subseteq X$ is totally bdd if $\forall \varepsilon > 0$, $\exists$ finitely many $B(x_i, \varepsilon)$, $x_i \in K$ $\ni$ $K \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon)$

(Meaning: paritition K into arbitrarily small parts)

Note: K totally bdd $\underset{\not\Leftarrow}{\Rightarrow}$ bdd

Pf: "$\Rightarrow$": Fix $\varepsilon > 0$

$\forall x, y \in K$, $x \in B(x_i, \varepsilon)$, $y \in B(x_j, \varepsilon)$

$\therefore \rho(x, y) \le \rho(x, x_i) + \rho(x_i, x_j) + \rho(x_j, y)$

$< \varepsilon + \max_{i,j} \rho(x_i, x_j) + \varepsilon$

$\Rightarrow \sup_{x,y \in K} \rho(x, y) \le 2\varepsilon + \max_{i,j} \rho(x_i, y_i) < \infty$

"$\not\Leftarrow$": K, X as in preceding example with $\varepsilon = \frac{1}{2}$

Thm (X, ρ) complete metric space

$K \subseteq X$ is compact $\Leftrightarrow$ K closed & totally bdd

Pf: "$\Rightarrow$":

$\forall x > 0$, $\{B(x, \varepsilon) : x \in K\}$ open covering of K

$\Rightarrow \exists$ $\{B(x_i, \varepsilon) : x_1, ..., x_n \in K\}$ covers K

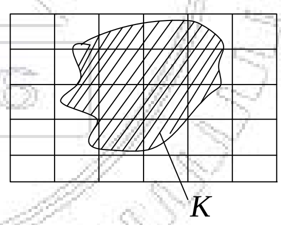
$\therefore K$ totally bdd

"$\Leftarrow$":

Let K be closed & totally bdd

Check: K sequentially compact

Let $\{y_n\} \subseteq K$



(1) K totally bdd

$\Rightarrow K \subseteq \bigcup_{i=1}^{n} B(x_i,1),\ x_i \in K$

$\Rightarrow\ \exists\ \{y_{n,1}\}$ of $\{y_n\}\ \ni\ \{y_{n,1}\} \subseteq B(x_i,1)$

$\uparrow$

pigeonhole principle

(2) $\because K \subseteq \bigcup_i B(z_i, \frac{1}{2}),\ z_i \in K$

$\Rightarrow \exists\ \{y_{n,2}\} \subseteq B(z_i, \frac{1}{2})$, subseq. of $\{y_{n,1}\}$

Consider the diagonal subseq. $\{y_{m,m}\}$ Cauchy seq.

$\because X$ complete

$\Rightarrow y_{m,m} \to y \in X$

$\because y_{m,m} \in K$ closed

$\Rightarrow y \in K$

$\therefore K$ sequentially compact

K totally bdd & infinite
$\Rightarrow \exists$ accumulation pt in X
(Bolzano-Weierstrass property)

Note: Similar to the proof of Bolzano-Weierstrass for $\mathbb{R}^n$

Note: In general, (X, ρ) complete metric space

$K \subseteq X$ bdd, infinite $\not\Rightarrow K$ has an accumulation pt in X

Ex. as before

But: (X, ρ) complete, $K \subseteq X$ totally bdd, infinite

$\Rightarrow k$ has accumulation pt

Homework: Ex. 3.5.2, 3.5.4, 3.5.5

Sec. 3.6 Continuous functions

Thm 1: (X, ρ) metric space

$Y \subseteq X$ compact

$f: Y \to \mathbb{R}$ conti.

$\Rightarrow f$ unif. conti.

Pf: (1) In advanced calculus, use definition of compactness (to replace Y by a finite set).

(2) Here, use sequen. compactness.

Assume otherwise.

$\therefore \exists \varepsilon > 0,\ \exists\, x_n, y_n$ in $Y\ \ni\ \rho(x_n, y_n) \to 0\ \&\ |f(x_n) - f(y_n)| \geq \varepsilon$

$\because Y$ sequentially compact

$\therefore \exists\, x_{n_k} \to x$ in Y & $\exists\, y_{n_k} \to y$ in Y

$$\therefore \rho(x,y) \le \underset{\substack{\downarrow \\ 0}}{\rho(x,x_{n_k})} + \underset{\substack{\downarrow \\ 0}}{\rho(x_{n_k},y_{n_k})} + \underset{\substack{\downarrow \\ 0}}{\rho(y_{n_k},y)}$$

$\Rightarrow \rho(x,y)=0$

$\Rightarrow x=y$

$$\therefore \underset{\substack{\vee/ \\ \varepsilon}}{\left|f(x_{n_k})-f(y_{n_k})\right|} \le \underset{\substack{\downarrow \\ 0}}{\left|f(x_{n_k})-f(x)\right|} \le \underset{\substack{\downarrow \\ 0}}{\left|f(y)-f(y_{n_k})\right|}$$

$\Rightarrow\ \to\leftarrow$

Thm 2. Same assumptions as above

$\Rightarrow \exists\, x_0, y_0 \in Y \ni f(x_0)=\sup_{x\in Y} f(x)$ & $f(y_0)=\inf_{x\in Y} f(x)$

In parti, f bdd

Pf: Let $M=\sup_{x\in Y} f(x) \le +\infty$

Let $x_n \in Y \ni f(x_n) \to M$

$\because Y$ sequentially compact

$\Rightarrow \exists\, x_{n_k} \ni x_{n_k} \to x_0 \in Y$

$\because f$ conti. on Y

$\Rightarrow M = f(x_0) < \infty$

Similarly for inf.

Tietze extension thm.

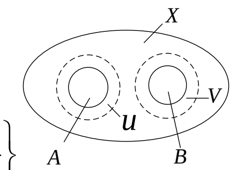
X Hausdorff top space

Then X normal $\Leftrightarrow \forall$ closed $Y \subseteq X$, $\forall$ conti. $f: Y \to \mathbb{R}$, $\exists$ extension to conti. $F: X \to \mathbb{R}$

Moreover, in this case, if f bdd, then F may be $\ni \inf_X F = \inf_Y f$, $\sup_X F = \sup_Y f$

Cor. If X is a metric space, then X has Tietze extension property.

Pf: "$\Leftarrow$": Check X is normal



Let $A, B \subseteq X$, closed & $A\cap B=\phi \Rightarrow d=d(A,B)>0$

Let $\mathcal{U}=\left\{x\in X:\ d(x,A)<\frac{d}{2}\right\}$, $V=\left\{x\in X:\ d(x,B)<\frac{d}{2}\right\}$

Then $\mathcal{U}, V$ open, $\mathcal{U}\cap V=\phi$ $\mathcal{U}\supseteq A$, $V\supseteq B$

$\Rightarrow X$ normal

(cf: J. Dugundji, Topology, pp.194-151)

X compact metric space

$C(X)=\{f: X\to \mathbb{R} \text{ conti.}\}$

$\rho(f,g)=\max_{x\in X}|f(x)-g(x)|$

Then $(C(X),\rho)$ complete metric space (as Ex.3.1.5)

Let $K\subseteq C(X)$

Question: When is K compact?

Answer: K compact $\Leftrightarrow K$ closed and totally bdd.

Quesiton: Can this condition be in terms of elements of K?

Answer:(Italian: Arzela & Ascoli)

K compact $\Leftrightarrow K$ closed & unif. bdd & equicontinuous

(X,ρ) metric space

Def: $\{f_\alpha\}$ on X is uniformly bdd if $\exists c>0 \ni |f_\alpha(x)|\le c \ \ \forall x\in X, \forall\alpha$

Def: $\{f_\alpha\}$ on X is equiconti. if $\forall\varepsilon>0, \exists\delta>0 \ni \rho(x,y)<\delta \Rightarrow |f_\alpha(x)-f_\alpha(y)|<\varepsilon \ \ \forall\alpha$

Note: Simultaneously unif. conti.

Thm. X compact metric space

$K\subseteq C(X)$

Then $\overline{K}$ compact $\Leftrightarrow K$ unif. bdd & equiconti.

Pf: "$\Rightarrow$": fix some $g_0\in\overline{K}$

Let $\overline{K}$ be compact

(1) $\because$ $\overline{K}$ is bdd in $C(X)\Rightarrow \rho(0,f)\le\rho(0,g_0)+\rho(g_0,f)$

$$\le \rho(0,g_0)+\sup_{f,g\in\overline{K}}\rho(f,g)<\infty \quad \forall f\in\overline{K}$$

$\Rightarrow K$ unif. bdd

(2) Check: K equi. conti.

$\because \overline{K}$ totally bdd

$\forall\varepsilon>0, \exists B(f_1,\varepsilon),\dots,B(f_n,\varepsilon) \ \ni K\in\overline{K}\subseteq\bigcup_{i=1}^{n}B(f_i,\varepsilon)$

$\therefore \forall f\in K, \exists f_i \ \ni |f(x)-f_i(x)|<\varepsilon \ \ \forall x\in X$

$$\begin{cases}\because \text{ Each } f_i \text{ is conti. on } X\\ \Rightarrow f_i \text{ unif. conti. on } X\\ \therefore \text{ for } \varepsilon>0, \exists\delta>0 \ni \rho(y,z)<\delta\Rightarrow|f_i(y)-f_i(z)|<\varepsilon\end{cases}$$

$\Rightarrow |f(y)-f(z)|\le|f(y)-f_i(y)|+|f_i(y)-f_i(z)|+|f_i(z)-f(z)|$

$<3\varepsilon \qquad \forall\rho(y,z)<\delta$

i.e. K equi. conti.

(Motivation: unif. conti. for finitely many $f_1,\dots f_n$ & totally bddness $\Rightarrow$ equi. conti. for K)

"$\Leftarrow$":

Arzela-Ascoli Lma:

$K \subseteq C(X)$ unif. bdd & equiconti., X compact metric space

$\{f_n\} \subseteq K$

Then $\exists f_{n_k} \to f \in C(X)$ in ρ

Pf: (1) Construction of f_{n_k} :

$\because X$ separable

Let $\{x_m\} \subseteq X$ dense

$\because \{f_n(x_1)\}$ bdd

$\Rightarrow \exists \{f_{n,1}(x_1)\}$ conv. (Bolzano-Weierstrass property for □)

$\because \{f_{n,1}(x_2)\}$ bdd

$\Rightarrow \exists \{f_{n,2}\}$ subseq of $\{f_{n,1}\}$ $\ni$ $\{f_{n,2}(x_2)\}$ conv.

$\vdots$

$\therefore \exists \{f_{n,k}(x_k)\}$ conv.

Let $g_n = f_{n,n}$

Then $\{g_n(x_k)\}$ conv. $\forall k$

(2) Unif. conv. of $\{g_n\}$:

Check: $\{g_n\}$ unif. Cauchy on X

$\because \{g_n\}$ also equiconti.

$\therefore \forall \varepsilon > 0,\ \exists \delta > 0 \ni \rho(x,y) < \delta \Rightarrow |g_n(x) - g_n(y)| < \varepsilon\ \forall n$

$\forall x \in X,\ |g_n(x) - g_m(x)| \le \underbrace{|g_n(x) - g_n(x_k)|}_{\wedge\ \varepsilon} + \underbrace{|g_n(x_k) - g_m(x_k)|}_{\wedge\ \varepsilon} + \underbrace{|g_m(x_k) - g_m(x)|}_{\wedge\ \varepsilon}$

if n, m large for such finitely many $\{x_k\}$

$\because X \subseteq \bigcup_{k=1}^{n} B(y_k, \frac{\delta}{2})$ (by compactness of X)	$\because x \in B(y_k, \frac{\delta}{2})$ for some k
$\{x_m\}$ dense in $X \Rightarrow \exists x_k \in B(y_k, \frac{\delta}{2})$	$\Rightarrow \rho(x,\ x_k) < \delta$

Let $f = \lim_n g_n$ pointwise $\Rightarrow g_n \to f$ unif. on X

$\uparrow$ as in (2)

Check: f conti.

$\because |f(x) - f(y)| \le |f(x) - g_n(x)| + |g_n(x) - g_n(y)| + |g_n(y) - f(y)|$

$\wedge$ $\wedge$ $\wedge$

ε ε ε

for large n by conti. of g_n

$\therefore f(x)$ is conti.

Pf : "$\Leftarrow$"(from K to $\overline{K}$)

Check: $\overline{K}$ sequen. compact

Let $\{f_n\} \subseteq \overline{K}$

Let $\{g_n\} \subseteq K \ni \rho(f_n, g_n) < \frac{1}{2^n} \ \forall n$

By Arzela-Ascoli, $\exists g_{n_k} \to f$ in $C(X)$ in ρ

$\therefore \rho(f_{n_k}, f) \le \rho(f_{n_k}, g_{n_k}) + \rho(g_{n_k}, f) < \frac{1}{2^{n_k}} + \varepsilon < 2\varepsilon$ for large k

$\Rightarrow f_{n_k} \to f$ in $C(X)$ in ρ

$\because f \in \overline{K} \Rightarrow \overline{K}$ sequen. compact

Cor. $K \subseteq \mathrm{R}^n$ compact $\Leftrightarrow K$ closed & bdd

Pf: Let $X = \{1, 2, ..., n\}$

$$\rho(x, y) = \begin{cases} 1 & \text{if } x \ne y \\ 0 & \text{if } x = y \end{cases}$$

$\therefore (X, \rho)$ compact metric space

$f : X \to \mathrm{R}$ conti. $\leftrightarrow x = \{x_1, ..., x_n\}$

$f(j) \leftrightarrow x_j$

$\therefore c(X) \leftrightarrow \mathrm{R}^n$

Homework: Ex. 3.6.10 (only for R)