Class 41

Sec. 37 Stone-Weierstiass Thm.

Weierstrass Thm.

$f:[a,b]\to R$ conti.

Then $\exists\ \{P_n\}\to f$ m$\|\cdot\|\infty$

Equivalently, poly's are dense in $(C[a,b],\|\ \|\infty)$, i.e., $\overline{P}=C[a,b]$

Proof of Weierstrass:

Lma. $L_n: C[a,b]\to C[a,b]\ \ni\ L_n(af+bg)=aL_n(f)+bL_n(g)\ \forall a,b\in R,\ f,g\in C[a,b]$.

$f\ge g\Rightarrow L_n f\ge L_n g\ \forall f,g\in C[a,b]$

Then $L_n f\to f$ unif. $\forall f\in C[a,b]$

$\Leftrightarrow L_n f\to f$ unif. for $f(x)=1,\ x,\ x^2$

Pf: E.W. Cheney, Introduction to approximation theory, pp. 67-68.

Pf. of Weierstrass (Bernstein, 1912): Assume $[a,b]\Rightarrow[0,1]$.

Let $(B_n f)(x)=\sum\limits_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$ for $f\in C[0,1],\ x\in[0,1],\ n\ge 1$.

Then B_n is linear & monotone.

$$\because (B_n 1)(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}=\left(x+(1-x)\right)^n=1$$

$$\begin{aligned}(B_n x)(x)&=\sum_{k=0}^{n}\frac{k}{n}\binom{n}{k}x^k(1-x)^{n-k}\\&=\sum_{k=1}^{n}\frac{k\cdot n!}{n(n-k)!k!}x^k(1-x)^{n-k}\\&=x\sum_{k=1}^{n}\binom{n-1}{k-1}x^{k-1}(1-x)^{n-k}\\&=x\sum_{j=0}^{n-1}\binom{n-1}{j}x^j(1-x)^{n-1-j}\quad(\text{let } j=k-1)\\&=x\left(x+(1-x)\right)^{n-1}=x\end{aligned}$$

$$\begin{aligned}\left(B_n x^2\right)(x)&=\sum_{k=0}^{n}\left(\frac{k}{n}\right)^2\binom{n}{k}x^k(1-x)^{n-k}\\&=\sum_{k=1}^{n}\frac{k}{n}\binom{n-1}{k-1}x^k(1-x)^{n-k}\end{aligned}$$

$$= \frac{n-1}{n} \sum_{k=2}^{n} \frac{k-1}{n-1} \binom{n-1}{k-1} x^k (1-x)^{n-k} + \frac{1}{n} \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k}$$

$$= \frac{n-1}{n} x^2 + \frac{1}{n} x \to x^2 \text{ unif. as } n \to \infty.$$

Pf. cf. K.M. Levasseur, Amer. Math. Monthly, 91(1984), 249-250

(use Chetyshev $\leq$ to give Bernstein's proof).

Let X compact Hausdorff space, $C(X) = \{ f : X \to R \text{ conti.}\}$

Let $\boldsymbol{a} \subseteq C(X)$ (replacing P)

(1) $\boldsymbol{a}$ is an algebra, i.e.,

$f, g \in \boldsymbol{a} \Rightarrow \underbrace{f+g,\ \lambda f}_{(\boldsymbol{a} \text{ vector space})}, f.g \in \boldsymbol{a}$
$\lambda \in R$

(2) $1(x) = 1 \ \forall x \in X$ is in $\boldsymbol{a}$

(3) $\boldsymbol{a}$ distinguishes pts. of X, i.e.,

$\forall\, x \neq y$ in X, $\exists\, f \in \boldsymbol{a} \to f(x) \neq f(y)$.

Note. P of $C[a,b]$ satisfies $(1),(2) \& (3)$.

Pf. for (3)

Let $x \neq y$ in $[a,b]$

Let $p(t) = \dfrac{t-x}{y-x}$ Then $p(x) = 0 \neq 1 = p(y)$

Stone-Weierstrass Thm.

X compact Hausdorff space

$\boldsymbol{a} \subseteq C(X)$ satisfies $(1),(2) \& (3)$.

Then $\overline{\boldsymbol{a}} = C(X)$

Note: If $\boldsymbol{a} \subseteq C(X)$ satisfies $(1) \& (2)$, not (3), then $\overline{\boldsymbol{a}}$ & $C(X)$ may not equal.

Ex. $\boldsymbol{a} = \{ f_\lambda : f_\lambda(x) = x \ \forall x \in X, \lambda \in \mathrm{R} \}$

Pf. (Assuming Weierstrass Thm)

Note: $\overline{\boldsymbol{a}}$ also satisfies (1), (2) & (3).

(1) $f \in \overline{\boldsymbol{a}} \Rightarrow |f| \in \overline{\boldsymbol{a}}$.

Reason:

Let $C_0 = \max_{x \in X} |f(x)|$. Then $f : X \to [-C_0, C_0]$.

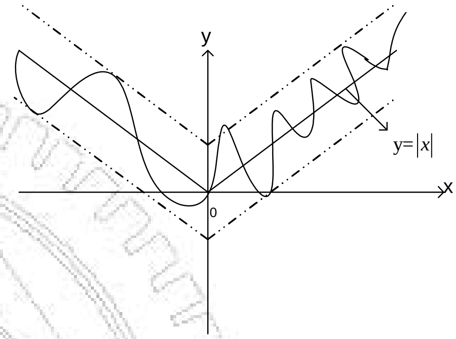
Weierstrass Thm for $[-C_0, C_0] \Rightarrow \exists$ poly. $p \ni \left||\lambda| - p(\lambda)\right| < \varepsilon \ \forall \lambda \in [-C_0, C_0]$.

Lma. 3.7.2. direct proof from advanced calculus

$\therefore \left||f(x)| - p(f(x))\right| < \varepsilon \ \forall x \in X$

$\because \overline{\boldsymbol{a}}$ algebra $\Rightarrow p \circ f \in \overline{\boldsymbol{a}}$

$\Rightarrow |f| \in \overline{\boldsymbol{a}}$



(2) $f, g \in \overline{\boldsymbol{a}} \Rightarrow \min(f, g), \max(f, g) \in \overline{\boldsymbol{a}}$

Reason: $\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g| \in \overline{\boldsymbol{a}}$

$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g| \in \overline{\boldsymbol{a}}$

(3) Let $f \in C(X)$, $x, y \in X$

Interpolate f by $f_{xy} \in \overline{\boldsymbol{a}}$ at x & y:

(i) $x = y$:

Let $f_{xy}(z) = f(x)$ $\forall z \in X$ const. func.

Then $f_{xy} \in \overline{\boldsymbol{a}}$ & $f_{xy} = f$ on x & y

(ii) $x \neq y$:

Then $\exists\, h \in \boldsymbol{a} \ni h(x) \neq h(y)$

Let $f_{xy}(z) = f(x) + (f(y) - f(x)) \cdot \dfrac{h(z) - h(x)}{h(y) - h(x)} \in \overline{\boldsymbol{a}}$

Then $f_{xy}(x) = f(x)$

$f_{xy}(y) = f(y)$

Interpolate f at x & approx f on X from below by $f_x \in \overline{\boldsymbol{a}}$:

(4) Fix $\varepsilon > 0$ & $x, y \in X$, $\because f = f_{xy}$ at x & both conti. $\Rightarrow f_{xy} < f + \varepsilon$ on a ball B_y

$\therefore \{B_y\}$ open covering of X

$\because X$ compact

$\Rightarrow \{B_{y_1}, \ldots, B_{y_m}\}$ covers X

Let $f_x = \min\{f_{xy_1}, \ldots, f_{xy_m}\} \in \overline{\boldsymbol{a}}$ (by (2) & (3))

Then $f_x = f$ at x

$\forall z \in X$, $z \in B_{y_j}$ for some j

$\Rightarrow f_x(z) \le f_{xy_j}(z) < f(z) + \varepsilon.$

i.e., $f_x < f + \varepsilon$ on X

(5) $\because f_x = f$ at x

$\Rightarrow f_x > f - \varepsilon$ at x

$\therefore \exists D_x \ni f_x > f - \varepsilon$ on D_x

$\therefore \{D_x\}$ open covering of X

$\Rightarrow \{D_{x_1}, \ldots, D_{x_n}\}$ covers X

Let $g = \max\{f_{x_1}, \ldots, f_{x_n}\} \in \overline{\boldsymbol{a}}$ (by (2) & (3))

$\forall z \in X$, $z \in D_{x_i}$ for some

$\Rightarrow g(z) \ge f_{x_i}(z) > f(z) - \varepsilon$

i.e., $g > f - \varepsilon$ on X

On the other hand, $g = \max\{f_{x_1}, \ldots, f_{x_n}\} < f + \varepsilon$ on X

i.e., $|f - g| < \varepsilon$ on X.

$\Rightarrow f \in \overline{\boldsymbol{a}}$

Generalization & Specialization:

(1) $X \subseteq \mathbb{R}^n$ compact

Let $\boldsymbol{a} = \{\text{poly. in coordinates}\}$

Then $\boldsymbol{a}$ satisfies (1), (2), (3)

$\Rightarrow \overline{\boldsymbol{a}} = \mathrm{C}(X)$ (Ex. 3.7.1)

Ex. $n = 2$:

$p(x_1, x_2) = x_1^3 x_2 - 3x_1 - 2x_2^2 + 1 \in \boldsymbol{a}$

$(2)\ \mathrm{C}(X)=\{f : X \to \mathbb{C} : \text{conti.}\}$: false as stated.

Ex. Let $\overline{\mathrm{X}}=\{z\in\mathbb{C} : |z|\le 1\}$ (cf. Ex.3.7.7)

Let $\boldsymbol{a}=\{\text{polynomials with complex coeffi.}\}$

Then $\boldsymbol{a}$ satisfies $(1),(2),(3)$

But $\overline{\boldsymbol{a}}=\{f \text{ conti. on } X \ \& \text{ analytic on Int}X\}\ne \mathrm{C}(X)$

Ex. $f(z)=\overline{z}\in\mathrm{C}(X)$ but not in $\overline{\boldsymbol{a}}$

Reason: In proof, min, max meaningless.