Class 42

(3) Stone-Weierstrass Thm for complex-valued func.:

X compact Hausdorff space

$\boldsymbol{a} \subseteq C(X)$ satisfies $(1),(2),(3)\,\&\,(4)$: $f \in \boldsymbol{a} \Rightarrow \overline{f} \in \boldsymbol{a}$

$\wedge$

$\lambda f \in \boldsymbol{a}\ \forall \lambda \in \mathbb{C}, f \in \boldsymbol{a}$

Then $\overline{\boldsymbol{a}} = C(X)$

Pf. Let $\boldsymbol{a}_1 = \{\text{real-valued func's in } \boldsymbol{a}\}$

$C_1(X) = \{\text{real-valued conti func's on } X\}$

Then $\boldsymbol{a}_1$ satisfies $(1),(2)\,\&\,(3)$

$\downarrow$

need (4)

$\Rightarrow \overline{\boldsymbol{a}}_1 = C_1(X)$

$\Rightarrow \overline{\boldsymbol{a}} = C(X)$

(Say, $x \neq y$ in X

$(3) \Rightarrow \exists f \in \boldsymbol{a} \ni f(x) \neq f(y)$

$\Rightarrow \operatorname{Re} f(x) \neq \operatorname{Re} f(y)$ or $\operatorname{Im} f(x) \neq \operatorname{Im} f(y)$

In any case, $\operatorname{Re} f, \operatorname{Im} f \in \boldsymbol{a}_1$

$\Downarrow$

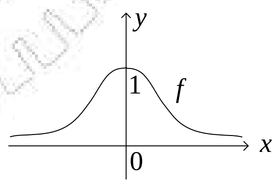
$(4) \Rightarrow \frac{1}{2}\left(f + \overline{f}\right) \in \boldsymbol{a}$)

Ex. $X = \{z \in \mathbb{C} : |z| \leq 1\}$

$\boldsymbol{a} = \left\{p(z) + \overline{q(z)} : p, q \text{ poly.}\right\}$ (trigonometric polynomials)

Then $\boldsymbol{a}$ satisfies $(1),(2),(3)\,\&\,(4)$

$\Rightarrow \overline{\boldsymbol{a}} = C(X)$



(4) $X \subseteq \mathbb{R}$ unbdd: false

Ex. $f(x) = e^{-x^2} \in C(\mathbb{R})$

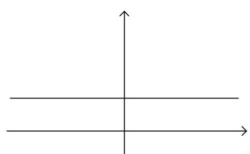
Assume $\{p_n\} \to \|f - p_n\|_\infty \to 0$ as $n \to \infty$.

Passing to subsequence if necessary, may assume p_n not const. $\forall n$

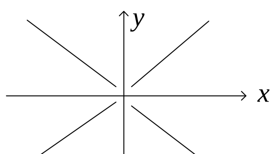
Then $|p_n(x)| \to \infty$ as $|x| \to \infty$

$|f(x)| \to 0$ as $|x| \to \infty$

$\Rightarrow \|f - p_n\|_\infty \not\to 0$



$\deg p_n = 0$



$\deg p_n \geq 1$

Homework: Ex. 3.7.2, Prove (3) above.

Sec. 3.8. Fixed-pt thm.

$(X,\rho),(Y,\sigma)$ metric spaces

Note: $f: X \to Y$ conti., X compact $\Rightarrow f$ unif. conti. (Ex. 3.8.1)

Reason: cf. the proof for $X=Y$

Thm. X, Y metric spaces, Y complete.

$X_0 \subseteq X$ dense

$f: X_0 \to Y$ unif. conti.

Then f can be extended uniquely to unif. conti. $\tilde{f}: X \to Y$.

Note 1. Applied for f linear map on vector spaces X, Y

Note 2. Thm is much easier than Tietze extension thm

Pf: $\forall x \in X \setminus X_0$, $\exists \{x_n\} \subseteq X_0 \ni x_n \to x$.

Define $\tilde{f}(x) = \lim_n f(x_n)$

Check: (1) limit exists

$\because \forall \varepsilon > 0, \exists \delta > 0 \ni \rho(x,y) < \delta \ x,y \in X_0 \Rightarrow \sigma(f(x), f(y)) < \varepsilon.$

$\therefore \{x_n\}$ Cauchy $\Rightarrow \exists N \ni n,m \ge N \Rightarrow \rho(x_n, x_m) < \delta$

$\Rightarrow \sigma(f(x_n), f(x_m)) < \varepsilon$

i.e., $\{f(x_n)\}$ Cauchy

$\because Y$ complete

$\Rightarrow \lim_n f(x_n)$ exists.

(2) $\tilde{f}$ well-defined, i.e., $\tilde{f}(x)$ indep. of $\{x_n\}$.

Say, $x_n \to x$, $y_n \to x$, where $x_n, y_n \in X_0$.

Let $f(x_n) \to z$, $f(y_n) \to w$.

Check: $z = w$

$$\because \sigma(z,w) \le \underset{\varepsilon}{\underset{\wedge}{\sigma(z, f(x_n))}} + \underset{\varepsilon}{\underset{\wedge}{\sigma(f(x_n), f(y_n))}} + \underset{\varepsilon}{\underset{\wedge}{\sigma(f(y_n), w)}}$$

$\uparrow$

For $\varepsilon > 0$, let δ be as before. $\because p(x_n, y_n) \le p(x_n, x) + p(x, y_n) < \delta$ for large n $\Rightarrow \sigma(f(x_n), f(y_n)) < \varepsilon$.

$\Rightarrow z = w$

(3) $\tilde{f}$ unif. conti. on X. (similar as (2) above)

(4) Let $g: X \to Y$ conti. & $g = f$ on X_0

Check: $g = \tilde{f}$ on X. (trivial)

Def. (X, ρ) metric space $T: X \to X$ is contraction

if $\exists \theta, 0 \le \theta < 1, \ni \rho(Tx, Ty) \le \theta\, \rho(x, y)\ \forall x, y \in X$

Note: T contraction $\Rightarrow T$ unif. conti.

(I) Thm. (Banach fixed-pt thm):

(metric fixed-pt thm) condi. on func.

(X, ρ) complete metric space.

$T: X \to X$ contraction ($\exists\ 0 \le \theta < 1 \ni \rho(Tx, Ty) \le \theta\rho(x, y)\ \forall x, y \in X$)

Then $\exists$ unique $z \in X \ni Tz = z$.

(II) Brouw & Schauder:

(top fixed-pt thm) condi. on domain

$K \subseteq \mathrm{R}^n$ compact, convex, nonempty.

$f: K \to K$ conti.

$\Rightarrow \exists x_0 \in K \ni f(x_0) = x_0$.

$\uparrow$

may not be unique

(III) Tarski's ordered fixed-pt thm

原型:

$f: [0,1] \to [0,1] \ni x \le f(x)\ \forall x \in [0,1]$

$\Rightarrow f$ has fixed-pt

Bourbaki fixed-point thm:

X partially ordered set (reflexive, anti-sym, transitive)

every chain of X has sup in X

$f: X \to X \ni x \le f(x)\ \forall x \in X$

$\Rightarrow f$ has fixed pt

Ex1. $f: [0,1] \to [0,1]$ conti.

$\Rightarrow f$ has fixed pt.

Ex2. $f: \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ conti. ($\mathbb{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$)

$\Rightarrow f$ has fixed pt.

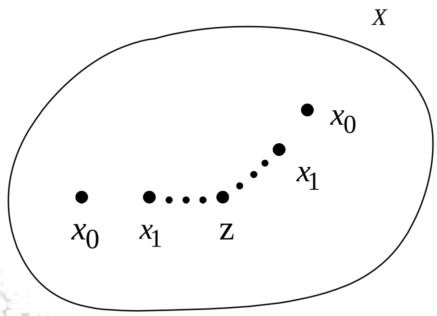
Note: In general, $\theta = 1$, false (cf. Ex. 3.8.5); false: $\rho(Tx, Ty) < \rho(x, y)\ \forall x, y$

Ex. $Tx = \ln\left(1+e^{x}\right): \mathbb{R} \to \mathbb{R}$

$$|Tx - Ty| = \frac{e^{x_0}}{1+e^{x_0}}|x-y| < |x-y| \text{ for some } x_0$$

If $Tz = z$, then $\ln\left(1+e^{z}\right) = z$

$\Rightarrow 1+e^{z} = e^{z} \to\leftarrow$



Pf: (1) Existence:

Fix $x_0 \in X$

Let $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$

Check: $\{x_n\}$ Cauchy

$$\rho(x_{n+1}, x_n) = \rho(Tx_n, Tx_{n-1}) \le \theta\rho(x_n, x_{n-1}) \le \ldots \le \theta^{n}\rho(x_1, x_0)$$

$$\Rightarrow \rho(x_m, x_n) \le \rho(x_m, x_{m-1}) + \ldots + \rho(x_{n+1}, x_n) \le \theta^{m-1}\rho(x_1, x_0) + \ldots + \theta^{n}\rho(x_1, x_0)$$

(say, $m > n$)

$$= \left(\theta^{m-1} + \ldots + \theta^{n}\right)\rho(x_1, x_0) = \theta^{n}\frac{1-\theta^{m-n}}{1-\theta}\cdot\rho(x_1, x_0) \le \theta^{n}\cdot\frac{\rho(x_1, x_0)}{1-\theta} \to 0 \text{ as } m, n \to \infty$$

$\therefore x_n \to z \in X$ $\boxed{\rho(z, x_n) \le \dfrac{\theta^{n}}{1-\theta}\rho(x_1, x_0)\ \forall n}$

↑ convergence rate: powers of θ

Check: $Tz = z$ $\quad\because$ T is unif. conti.

$\therefore x_{n+1} = Tx_n$ as $n \to \infty$

$\downarrow$ z $\qquad \downarrow$ Tz

(2) Uniqueness:

Assume $Ty = y$ & $Tz = z$

Check: $y = z$

$\because \rho(y, z) = \rho(Ty, Tz) \le \theta\rho(y, z)$

If $\rho(y, z) \ne 0$, then $1 \le \theta \to\leftarrow$

$\Rightarrow \rho(y, z) = 0$

$\Rightarrow y = z$

Applications of Banach:

(1) O.D.E. with initial condi. ;

(2) integral equa (Ex. 3.8.3);

(3) implicit func. thm (cf. J.Dugundji, pp. 306-307). (due to T.H.Hildebrat & L.M.Graves, 1927)

(4) inverse func. thm (cf. W.Rudin, Principle of math analysis, 3rd ed., p.221)

(5) Newton's method

(6) cobweb thm

(7) Fundamental thm of Markov chains

(8) Jacobis method

(9) Gauss-Seidel method

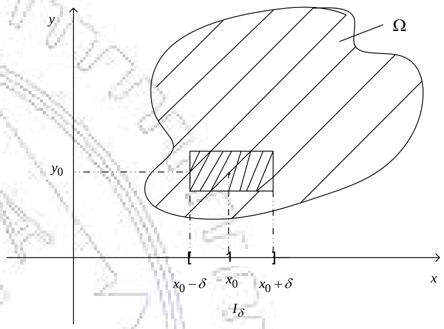
(cf. C.H.Wagner, A generic approach to iterative methods, Math. Mag., 55 (1982), 259-273)

Initial value problem:

Assume $f:\Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$, $(x_0, y_0) \in \Omega$

open

$$\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$$



Def. $y(x): I_\delta \to \mathbb{R}$ is a solu. if

$$\begin{cases} (1)\ y \in C'(I_\delta); \\ (2)\ (x, y(x)) \in \Omega \ \ \forall x \in I_\delta; \text{ (so that (3) is meaningful)} \end{cases}$$

$$\begin{cases} (3)\ y'(x) = f(x, y(x)) \ \forall x \in I_\delta; \\ (4)\ y(x_0) = y_0 \end{cases}$$

Moral:
(1) Banach $\Rightarrow$ Picard
(2) Brouwer $\Rightarrow$ Peano

Thm. (Picard)

f conti., bdd on Ω

f Lipschitz w.r.t. y in Ω.

i.e., $\exists K > 0 \ni \left| f(x_1, y_1) - f(x_1, y_2) \right| \le K \cdot |y_1 - y_2| \ \forall (x_1,y_1),(x_1,y_2) \in \Omega$

Then $\exists$ unique solu. y in some nbd I_δ of x_0