Class 43

Sketch of proof:

Let $M = \sup_{\Omega} |f|$

Let δ be $\ni$ $\delta < \frac{1}{K}$ & $[x_0-\delta, x_0+\delta] \times [y_0 - M_\delta, y_0 + M_\delta] \subseteq \Omega$

Let $X = \{\phi \in C(I_\delta) : \phi(x_0) = y_0$ & $\phi(x) \in [y_0 - M_\delta, y_0 + M_\delta] \forall x \in I_\delta\}$.

Let $T : X \to C(I_\delta)$ be $\ni$

$(T\phi)(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt$ for $x \in I_\delta$.

Check: (1) X closed in $C(I_\delta)$;

$\Rightarrow X$ complete metric space

(2) $TX \subseteq X$;

(3) T contraction

Hence $\exists 1 \phi \in X \ni T\phi = \phi$

$\therefore \phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt \ \forall x \in I_\delta$

$\Rightarrow \frac{d\phi}{dx} = f(x, \phi(x))$

Note1. Essentially, starting with $\phi(x) = y_0$ then seccessively approximate to the solu.

Note 2. If f only conti., bdd on Ω, then $\exists$ solu. (Peano's thm)

But not necessarily unique. (can be proved by Brouwer)

Ex. $x_0 = y_0 = 0$ & $f(x,y) = 3y^{\frac{2}{3}}$ on $\Omega = (-1,1) \times (-1,1)$

i.e., $y' = 3y^{\frac{2}{3}}, y(0) = 0$

Then two solu's: $y_1 \equiv 0$ & $y_2 = x^3$

Explanation: $\frac{f(0,y) - f(0,0)}{y - 0} = 3y^{-\frac{1}{3}}$ unbdd on Ω.

Homework:

Ex.3.8.4. (use only Banach, not Brouwer); 3.8.5