Class 44

Chap.4. Banach spaces

Functional analysis:

Consider spaces of functions.

topology + algebra

(1) space & operator: duality theory

Reason: In C^n, inner product
In Hilbert space, $(\ , \ )$
In Banach space B, $\| \ \|$
But (B,B*), $(x, f) = f(x)$

(2) operator: spectral theory (compact, normal)

spectrum-eigenvalue.

$(T-\lambda I)(x) = y$ or $Tx = y$

$$X \begin{cases} \text{real vector space} & : +, . \text{ over R} \\ \text{complex vector space} & : +, . \text{ over C} \end{cases}$$

Let F = R or C

$\rightarrow$ independence, span, basis, dimension, normed space:

$$\| \ \|: \begin{cases} x \mapsto \|x\| : X \to \mathrm{R} \ \ni \\ (1) \ \|x\| \geq 0 \ \forall x, \\ (2) \ \|x\| = 0 \Leftrightarrow x = 0, \\ (3) \|\lambda x\| = |\lambda| . \|x\|, \\ (4) \|x+y\| \leq \|x\| + \|y\| \end{cases}$$

$\Rightarrow \rho(x, y) = \|x - y\|$ metric

Def. Banach space: (X, ρ) complete

Def. X metric linear space if

(1) X vector space,

(2) X metric space with ρ,

(3) $(x, y) \mapsto x + y$ ... $X \times X \to X$

$(\lambda, x) \mapsto \lambda x$ are conti. from $F \times X \to X$

$\Downarrow$

R or C

Def. X Fre'chet space if

(1) X metric linear space;

(2) $\rho(x, y) = \rho(x+z, y+z), \forall x, y, z \in X$

(3) X complete.

Zorn's Lemma.

S partially ordered set

(i.e., reflexive & transitive & anti-symmetric)

$\forall\ T \subseteq S$ totally ordered, $\exists$ upper bd (in S)

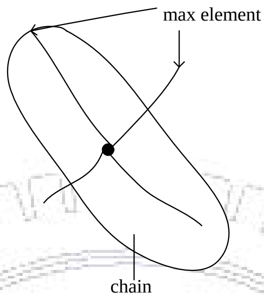
$\uparrow$ $\qquad\qquad$ $\uparrow$

(i.e., all pairs comparable) (i.e., larger than every element in T)

$\Rightarrow S$ has a max element, say, y

(i.e., $x \le y \Rightarrow y \le x$)

Ex.



Thm X vector space

$\Rightarrow X$ has a linearly independent spanning set a

Def. a Hamel basis for X

note: $\forall x \in X$, $x = \sum_{i=1}^{n} \lambda_i y_i$, $\lambda_i \in F$, $y_i \in a$

Pf: partially order the collection of indep subsets of X.

apply Zorn's Lemma

Note 1: Banach space $\underset{\not\Leftarrow}{\Rightarrow}$ Fre′chet space

$\downarrow$ $\qquad\qquad$ $\downarrow$

metric: scale-invariant $\qquad$ metric: translation-invariant

i.e., $\rho(ax, ay) \ne |a|\rho(x, y)$

Ex. □ with $\rho(x,y) = \dfrac{|x-y|}{1+|x-y|}$

Then Fre′chet, not Banach

$\uparrow$

Reason: If $\rho(x,y) = \|x-y\|$ for some $\|\cdot\|$,

then $\|2(x-y)\| = 2\|x-y\| = 2\rho(x,y)$

$\|$ $\qquad\qquad$ $\|$

$\rho(2x, 2y)$

$\|$ $\qquad\qquad$ $2.\dfrac{|x-y|}{1+|x-y|}$

$\dfrac{2|x-y|}{1+2|x-y|}$

Let $x = 1, y = 0 \Rightarrow \ \rightarrow\leftarrow$

Note 2: Every normed space X can be embedded in a Banach space $\tilde{X}$

Note. Q ~> $\tilde{X}$ can be constructed as this.

i.e., X isometrically isomorphic to a dense subset of $\tilde{X}$ & $\tilde{X}$ unique

Pf: $X \subseteq \tilde{X}$, complete metric space

$\{\tilde{x} =$ equivalence class of $\{x_n\}$. Cauchy sequ.$\}$

$\{x_n\}$, $\{y_n\}$ Cauchy.

Def. $\{x_n\}$, $\{y_n\}$ equiv. if $\lim_n \|x_n - y_n\| = 0$

Define $\tilde{x} + \tilde{y} = \tilde{x}\{x_n + y_n\}$

$\lambda\tilde{x} = \{\lambda x_n\}$

$\|\tilde{x}\| = \lim \|x_n\|$. Then $\tilde{X}$ Banach space & $X \cong \{\{x, x, \ldots\} : x \in X\}$ etc.

X normed space

Def. $\sum_n x_n$ converges, absolutely conv.

Thm. X normed space

Then X Banach space iff every abso. conv. series is conv. (Ex. 4.1.6)

Pf: "$\Rightarrow$":

Assume $\sum_n x_n$ abso. conv.

Let $s_n = \sum_{j=1}^{n} x_j$

Then $\|s_n - s_m\| \underset{(n<m)}{=} \left\| \sum_{j=n+1}^{m} x_j \right\| \le \sum_{j=n+1}^{m} \|x_j\| \to 0$ as $n < m \to \infty$.

$\because$ X Banach space

$\Rightarrow \{s_n\}$ converges in X.

"$\Leftarrow$":

Let $\{y_n\}$ be Cauchy. Choose $\{y_{n_k}\}$ $\ni$ $\sum_k \|y_{n_{k+1}} - y_{n_k}\| < \infty$ as follows:

For $\varepsilon = 1$, let n_1 be $\ni$ $i, j \ge n_1 \Rightarrow \|y_i - y_j\| < 1$ $\qquad \Rightarrow \|y_{n_2} - y_{n_1}\| < 1$

$\vdots$

For $\varepsilon = \frac{1}{2}$, let n_2 be $\ni$ $n_2 > n_1$ & $i, j \ge n_2 \Rightarrow \|y_i - y_j\| < \frac{1}{2} \Rightarrow \|y_{n_3} - y_{n_2}\| < \frac{1}{2}$

$\vdots$

For $\varepsilon = \frac{1}{4}$, let n_3 be $\ni$ $n_3 > n_2$ & $i, j \ge n_3 \Rightarrow \|y_i - y_j\| < \frac{1}{4}$

$\vdots$

Then $\|y_{n_{k+1}} - y_{n_k}\| < \frac{1}{2^{k-1}}$ $\quad \forall$k

Let $x_1 = y_{n_1}$

$x_k = y_{n_k} - y_{n_{k-1}}$ for $k \geq 2$

Then $\sum_k \|x_k\| < \infty$

$\Rightarrow \sum_k x_k$ converges

i.e., partial sum $= y_{n_k}$ converges, say, to y.

Then $\|y_n - y\| \leq \underset{\varepsilon}{\underset{\wedge}{\|y_n - y_{n_k}\|}} + \underset{\varepsilon}{\underset{\wedge}{\|y_{n_k} - y\|}}$

i.e., y_n converges to y

$\therefore X$ Banach space

Ex. $\left.\begin{array}{l} X = \{\text{poly. on } [0,1]\} \\ \|p\| = \max_{x\in[0,1]} |p(x)| \end{array}\right\}$ normed space

Then $\exists$ abso. conv. series not conv. in X

Def. X vector space

$\|\cdot\|_1, \|\cdot\|_2$ norms are equivalent if $\exists\, a, b > 0 \ni a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \ \forall x \in X$.

$\downarrow$

Meaning same top, but different norms.

Note: 1. X infinite-dim $\Rightarrow \exists\ \|\cdot\|_1, \|\cdot\|_2$ not equiv. (Ex. 4.2.6)

2. X finite-dim $\Rightarrow \forall\ \|\cdot\|_1, \|\cdot\|_2$ are equiv. (Ex. 4.3.1)

In other words, $\dim X < \infty \Leftrightarrow$ all norms on X are equiv.

(alg. condi.) (top. condi.)

Pf. of note 1:

Let $\{x_\alpha\}$ Harmel basis of X.

$\forall x \in X, x = \sum \lambda_\alpha x_\alpha$, where $\lambda_\alpha = 0$ for all but finitely many α's.

Define $\|x\|_1 = \sum_\alpha |\lambda_\alpha| a_\alpha$, where $a_\alpha > 0\ \forall \alpha$

$\|x\|_2 = \sum_\alpha |\lambda_\alpha| b_\alpha$, where $b_\alpha > 0\ \forall \alpha$

Then both norms.

If $\|\cdot\|_1 \sim \|\cdot\|_2$, then $a \cdot \sum_{\alpha} |\lambda_\alpha| a_\alpha \leq \sum_{\alpha} |\lambda_\alpha| b_\alpha$

Let $x = x_\alpha \Rightarrow \lambda_\alpha = 1$ & $\lambda_\beta = 0\ \forall \beta \neq \alpha$

In parti., $a \cdot a_\alpha \leq b_\alpha\ \ \forall \alpha \Rightarrow a \leq \dfrac{b_\alpha}{a_\alpha} \forall \alpha$

Let $\dfrac{b_\alpha}{a_\alpha} \to 0$. Then $a = 0\ \ \to\leftarrow$

Banach spaces:

Ex.1. $L^P(X,u)$ with $\|\cdot\|_p, \|\cdot\|_\infty\ (1 \leq p \leq \infty)$

func.'s a.e. are identified.

Ex.2. $l^P\ (1 \leq p \leq \infty)$ (i.e., $u =$ counting measure of $\{1,2,3,...\}$).

Ex.3 $C(X)$ with $\|\ \|_\infty$

X compact (metric) space.

New spaces from old:

Def. (1) $X_1, ..., X_n$ vector spaces

Let $X = \sum_{i=1}^{n} \oplus X_i$ or $X_1 \times ... \times X_n$ be $\{(x_1, ..., x_n):\ x_i \in X_i\}$

Define addition and scalar product componentwise

Then X vector space

(2) $(X_1, \|\cdot\|_1), ..., (X_n, \|\cdot\|_n)$ normed spaces

Define $\|(x_1, ..., x_n)\| = \begin{cases} (\sum_{i=1}^{n} \|x_i\|_i^p)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \max_i \|x_i\|_i & \text{if } p = \infty \end{cases}$,

Then equivalent norms & $(X, \|\cdot\|)$ normed space

(3) $X_1, ..., X_n$ Banach spaces $\Rightarrow X_1 \times ... \times X_n$ Banach space (Ex. 4.1.5)

Homework: Ex. 4.1.3., 4.1.4, 4.1.5

Sec. 4.2 Subspace & bases

X over F=R or C

Def. Subspace of vector space

Def. Subspace spanned (generated) by a subset $K \subseteq X$

$\Downarrow$

$\{\lambda_1 x_1 + ... + \lambda_n x_n : \lambda_i \in F, x_i \in K\}$ (note: K may be infinite)

Note: D (normed space) closed subspace (spanned by K)

i.e., $\overline{\{\lambda_1 x_1 + ... + \lambda_n x_n :\ \lambda_1, ..., \lambda_n \in F, x_1, ..., x_n \in K\}}$