Class 45-46

Quotient space

X normed space

$Y_0 \subseteq X$ closed subspace

Let $X/Y_0 = \{x+Y_0 : x \in X\}$ $(=\{$equivalence classes of elements of X under $x \equiv y$ if $x-y \in Y_0\})$

Define $(x_1+Y_0)+(x_2+Y_0) \equiv (x_1+x_2)+Y_0$

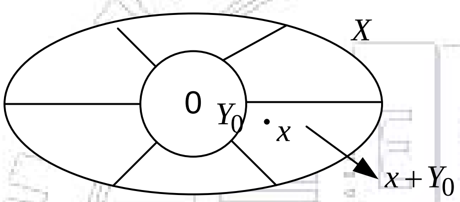
$\lambda(x+Y_0) \equiv \lambda x + Y_0$

Then X/Y_0 vector space.

Define $\|x+Y_0\| = \inf_{y \in Y_0} \|x+y\|$

Thm, $(X/Y_0, \|\ \|)$ normed space

Pf.:



(1) $\|Y_0\| = \inf_{y \in Y_0} \|y\| = 0$

Conversely, if $\|x+Y_0\| = \inf_{y \in Y_0} \|x+y\| = 0$, then

$\exists\, y_n \in Y_0 \ni \|x+y_n\| \to 0$,

i.e., $y_n \to -x$ in $\|\ \|$ $\because Y_0$ closed $\Rightarrow x \in Y_0$

$\Rightarrow x+Y_0 = Y_0$

(2) $\|\lambda(x+Y_0)\| = \|\lambda x + Y_0\| = \inf_{y \in Y_0} \|\lambda x + y\|$

$= \inf_{y_1 \in Y_0} \|\lambda x + \lambda y_1\| = |\lambda| \inf_{y_1 \in Y_0} \|x+y_1\| = |\lambda| \|x+Y_0\|$

(3) $\|(x_1+Y_0)+(x_2+Y_0)\| - \varepsilon = \|(x_1+x_2)+Y_0\| - \varepsilon$

$= \inf_{y \in Y_0} \|x_1+x_2+y\| - \varepsilon$

$= \inf_{y_1, y_2 \in Y_0} \|x_1+x_2+y_1+y_2\| - \varepsilon$

$\wedge\!\backslash$

$\|x_1+y_1\| - \frac{\varepsilon}{2} + \|x_2+y_2\| - \frac{\varepsilon}{2}$

$\exists\, y_1, y_2 \in Y_0 \ni$

$\le \inf_{y_1 \in Y_0} \|x_1+y_1\| + \inf_{y_2 \in Y_0} \|x_2+y_2\| = \|x_1+Y_0\| + \|x_2+Y_0\|$. Then let $\varepsilon \to 0$

Note: (Ex. 4.2.2) X Banach space $\Rightarrow X/Y_0$ Banach space

Homework: Ex. 4.2.2, 4.2.4

Thm. X normed space over $F = \mathbb{R}$ or $\mathbb{C}$

If $\underset{n}{\underset{\|}{\dim}} X < \infty$, then X homeom, isomorphic to $F^{(n)}$

Pf. Let $\{e_1, \ldots, e_n\}$ be a Hamel basis for X.

$\forall x \in X, x = \sum_{i=1}^{n} \lambda_i e_i$ uniquely.

Then $\tau : x \mapsto (\lambda_1, \ldots, \lambda_n)$, 1-1, onto, isomorphism: $X \to F^{(n)}$

Check: homeom.

(1) Check: τ^{-1} conti.

Assume $\left(\lambda_1^{(m)}, \ldots, \lambda_n^{(m)}\right) \to (\lambda_1, \ldots, \lambda_n)$

Then $\left\|x^{(m)} - x\right\| = \left\|\sum_i \lambda_i^{(m)} e_i - \sum_i \lambda_i e_i\right\| \le \sum_i \left|\lambda_i^{(m)} - \lambda_i\right| \cdot \|e_i\|$

$\le M \cdot \sum_i \left|\lambda_i^{(m)} - \lambda_i\right| \to 0$

(2) Check: τ conti.

$\because (1) \Rightarrow \tau^{-1}$ conti.

$\therefore \tau^{-1}\left(\underbrace{\left\{(\lambda_1, \ldots, \lambda_n) : \sum_j |\lambda_j| = 1\right\}}_{\text{compact}}\right) = \left\{\sum_j \lambda_j e_j : \sum_j |\lambda_j| = 1\right\}$ compact

Let $C = \inf\left\{\left\|\sum_j \lambda_j e_j\right\| : \sum_j |\lambda_j| = 1\right\} = \left\|\sum_j \tilde{\lambda}_j e_j\right\|$ for some $\sum_j |\tilde{\lambda}_j| = 1$

If $C = 0$, then $\sum_j \tilde{\lambda}_j e_j = 0$

$\Rightarrow \tilde{\lambda}_j = 0 \ \forall j \quad \to\leftarrow \quad \Rightarrow C > 0$

$\therefore \left\|\sum_j \lambda_j e_j\right\| \ge C \cdot \sum_j |\lambda_j| \ \forall \lambda_j \in F$

$\Rightarrow \tau$ conti.

Note. In parti., $C \cdot \sum_j |\lambda_j| \le \left\|\sum_j \lambda_j e_j\right\| \le M \cdot \sum_j |\lambda_j| \ \forall \lambda_j \in F$

i.e., $(X, \|\cdot\|) \sim (F^{(n)}, \|\cdot\|_1)$

Cor 1.(cf. Ex. 4.3.1) X finite-dim vector space, $\|\cdot\|_1 \|\cdot\|_2$ norms on X.

Then $\|\cdot\|_1 \sim \|\cdot\|_2$.

i.e., $\exists\, a,b>0 \ni a\|x\|_1 \le \|x\|_1 \le b\|x\|_2 \ \forall x \in X$

Pf.: By note of thm, $(X,\|\cdot\|_1) \sim (F^{(n)},\|\cdot\|_1) \sim (X,\|\cdot\|_2)$

Cor 2. X normed space over F

$Y \subseteq X$ finite-dim subspace

$\Rightarrow Y$ closed.

Pf. Let $\{y_m\} \subseteq Y \ni y_m \to y \in X$

Check: $y \in Y$

$\because \{y_m\}$ bdd in finite-dim $Y \Rightarrow \left\{\left(\lambda_1^{(m)},\ldots,\lambda_n^{(m)}\right)\right\}$ bdd in $F^{(n)}$ (by note of thm)

$\therefore$ Bolzano-Weierstrass $\Rightarrow$ subseq. conv.

$\Rightarrow \exists \{y_{m'}\}$ converges to $z \in Y$

But $y_{m'} \to y$

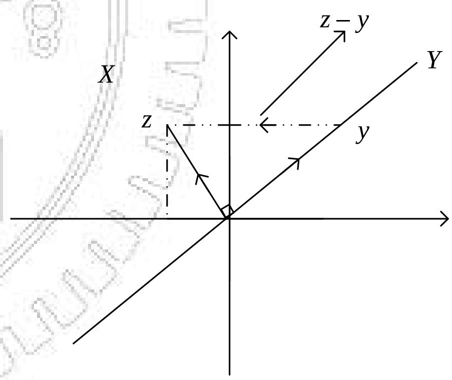
$\therefore y = z \in Y$

Cor. 3 X finite-dim normed space $\Rightarrow X$ Banach space

Pf: $\because X$ dense in Banach space $\tilde{X}$

Cor. 2 $\Rightarrow X$ closed $\Rightarrow X = \overline{X} = \tilde{X}$ is a Banach space

For the proof of next thm, need the following lemma.



Riesz's Lemma:

X normed space

$Y \underset{\neq}{\subset} X$ closed subspace

Then $\forall \varepsilon>0, \exists\, z \in X \ni \|z\|=1$ & $\|z-y\| > 1-\varepsilon \ \ \forall y \in Y$

Explanation: In normed space, $\exists$ elements in $Y^{\perp}$ approximately

Let $z \perp Y$ & $\|z\|=1$
Then $\|z-y\| \ge \|z\| = 1 > 1-\varepsilon \ \ \forall y \in Y$

Pf.: Let $x_0 \in X$, $x_0 \notin Y$

Let $d = \inf_{y \in Y} \|x_0 - y\|$.

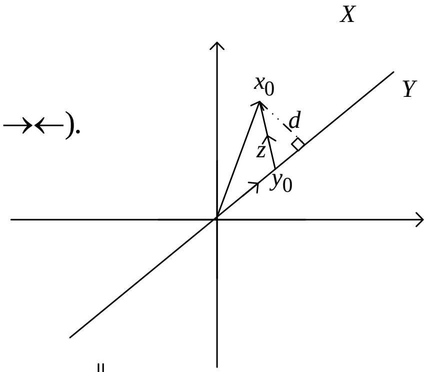
Then $d > 0$

(If $d = 0$, then $\exists\, y_n \in Y \ni \|x_0 - y_n\| \to 0 \Rightarrow x_0 \in Y \to\leftarrow$).

$\forall \eta > 0$, $\exists\, y_0 \in Y \ni d \le \|x_0 - y_0\| \le d + \eta$

Let $z = \dfrac{x_0 - y_0}{\|x_0 - y_0\|}$

Then $\|z\| = 1$



$$\forall y \in Y,\ \|z - y\| = \left\| \frac{x_0 - y_0}{\|x_0 - y_0\|} - y \right\| = \frac{1}{\|x_0 - y_0\|} \cdot \left\| x_0 - \underbrace{\left(y_0 + \|x_0 - y_0\| y\right)}_{\substack{\Downarrow \\ \in Y}} \right\|$$

$$\ge \frac{d}{d + \eta} > 1 - \varepsilon$$

(let $0 < \eta < \dfrac{d\varepsilon}{1 - \varepsilon}$)

Thm. X normed space over F, "$K \subseteq X$ closed & bdd $\Rightarrow$ compact" $\Leftrightarrow$ $\dim X < \infty$.

Pf: note: top condi. $\Leftrightarrow$ algebric condi.

"$\Leftarrow$": by Thm before

"$\Rightarrow$":

Assume $\dim X = \infty$.

(Idea: construct approx. o.n. sequence $\{x_n\}$ by Riesz's Lemma)

Motivating example: $X = l^2, e_n = (0, \ldots, 0, 1, 0, \ldots), n \ge 1$

$K = \overline{\{e_n : n \ge 1\}}$ $\quad n$th

Then K closed & bdd, but $\|e_n - e_m\| = \sqrt{2}\ \forall n \ne m$

$\Rightarrow \{e_n\}$ not Cauchy

$\Rightarrow \{e_n\}$ has no conv. subseq.

$\Rightarrow K$ not sequentially compact

$\Rightarrow K$ not compact

Let $x_1 \in X \ni \|x_1\| = 1$

Let $Y = \langle x_1 \rangle \underset{\neq}{\subset} X$ (Y finite-dim $\Rightarrow Y$ closed)

Riesz's Lemma for $\varepsilon = \frac{1}{2} \Rightarrow \exists\, x_2 \in X,\ \|x_2\| = 1 \ni \|x_2 - x_1\| > 1 - \frac{1}{2} = \frac{1}{2}$

Let $Y = \langle x_1, x_2 \rangle \underset{\neq}{\subset} X$ (Y finite-dim $\Rightarrow Y$ closed)

Riesz's Lemma for $\varepsilon = \frac{1}{2} \Rightarrow \exists\, x_3 \in X,\ \|x_3\| = 1$ & $\|x_3 - x_1\|,\ \|x_3 - x_2\| > 1 - \frac{1}{2} = \frac{1}{2}$

......

$\Rightarrow \exists \{x_n\} \subseteq X \ni \|x_n\| = 1$ & $\|x_n - x_j\| > \frac{1}{2}\ \forall\ 1 \le j < n.$

$\therefore \{x_n\}$ bdd, but no subseq. conv. ($\because$ Cauchy not satisfied)

i.e., $\overline{\{x_n\}}$ not sequentially compact, but closed, bdd.

$\Rightarrow \overline{\{x_n\}}$ not compact. $\rightarrow\leftarrow$

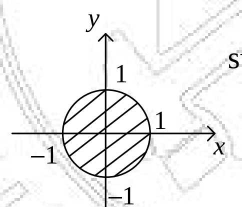
Homework:

Sec. 4.3.

Ex. 4.3.2~ 4.3.4

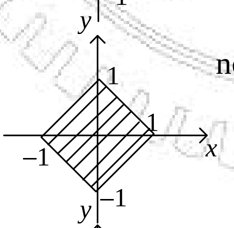
$\|\cdot\|_p$ strictly convex if $1 < p < \infty$; otherwise if $p = 1, \infty$.

Ex. 1. $(\mathrm{R}^2, \|\cdot\|_2)$:



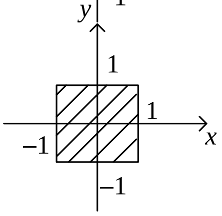
strictly convex

Ex. 2. $(\mathrm{R}^2, \|\cdot\|_1)$:



not strictly convex

Ex. 3. $(\mathrm{R}^2, \|\cdot\|_\infty)$:



not strictly convex

Sec. 4.4. Linear Transformations.

Thm.1. X, Y normed spaces.

$T: X \to Y$ linear transf.

(i.e., $T(x+y) = Tx + Ty \ \forall x, y \in X$ & $T(\lambda x) = \lambda Tx \ \forall \lambda \in F, x \in X$)

Then T conti. on $X \Leftrightarrow T$ conti. at one pt. of X

Ex.1. $F \to F$

$x \mapsto ax$

Ex.2. $F^{(n)} \to F^{(n)}$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \to \begin{bmatrix} t_{11} \cdots t_{1n} \\ \vdots \quad \vdots \\ t_{n1} \cdots t_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Pf.:"$\Rightarrow$": Trivial

"$\Leftarrow$":

Assume T conti. at $z \in X$

Let $x \in X$

Check: T conti. at $x \in X$

Assume $x_n \to x$

Check: $Tx_n \to Tx$

$\because x_n \to x \Rightarrow x_n - x + z \to z$

$$\Rightarrow T(x_n - x + z) \to Tz$$
$$\parallel$$
$$Tx_n - Tx + Tz$$

$\Rightarrow Tx_n \to Tx$ as needed.

Thm 2. X, Y normed spaces.

$T: X \to Y$ linear transf.

Then T conti. $\Leftrightarrow \sup\limits_{x \neq 0} \dfrac{\|Tx\|}{\|x\|} < \infty$

Pf: "$\Rightarrow$"

Assume $\sup\limits_{x \neq 0} \dfrac{\|Tx\|}{\|x\|} = \infty$

$\therefore \forall n \Rightarrow \exists\, x_n \neq 0 \ni \dfrac{\|Tx_n\|}{\|x_n\|} \geq n$

$$\therefore \left\| T\left(\frac{x_n}{n\|x_n\|} \right) \right\| \geq 1$$
$$|||$$
$$y_n$$

Then $y_n \to 0$ ($\because \|y_n\| = \frac{1}{n} \to 0$)

But $Ty_n \not\to 0$ $\quad\to\leftarrow$

"$\Leftarrow$": Let $M = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} < \infty$

Check: T conti. at 0

Let $x_n \to 0$

$\because \frac{\|Tx\|}{\|x\|} \le M \quad \forall x \neq 0$

$\Rightarrow \|Tx\| \le M \cdot \|x\| \quad \forall x$

$\therefore \|Tx_n\| \le M \cdot \|x_n\| \to 0$

$\Rightarrow Tx_n \to 0$, i.e., T conti. at 0.

Note: X, Y normed spaces over F, $\dim X < \infty$

$T: X \to Y$ linear transf. $\Rightarrow$ T conti. (Homework)

Def. T conti. linear transf.: $X \to Y$

$$\|T\|_{X,Y} = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} \text{ (norm of } T\text{)}$$

Note 1. $\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\| \le 1} \|Tx\| = \sup_{0 < \|x\| \le 1} \neq \frac{\|Tx\|}{\|x\|}$

2. $\|Tx\| \le \|T\| \cdot \|x\| \quad \forall x \in X$

3. T conti. at one pt. of $X \Leftrightarrow T$ conti. on $X \Leftrightarrow T$ unif. conti. on X

Ex. $T \cdot \mathbb{C}^n \to \mathbb{C}^n, T = [a_{ij}]$

(1) $\|T\|_1 = \max_{x \neq 0} \frac{\|Tx\|_1}{\|x\|_1} = \max_j \sum_i |a_{ij}|$ (max column sum)

(2) $\|T\|_\infty = \max_{x \neq 0} \frac{\|Tx\|_\infty}{\|x\|_\infty} = \max_i \sum_j |a_{ij}|$ (max row sum)

(3) $\|T\|_2 = \max_{x \neq 0} \frac{\|Tx\|_2}{\|x\|_2}$ = max singular value of T (cf. Ex.4.4.3)