Class 47

Thm. X,Y normed spaces.

$B(X,Y)=\{\text{conti. linear transf. } T:X\to Y\}$

Then $\left(B(X,Y),\|\cdot\|\right)$ normed space

Pf.: Let $T\in B(X,Y)$

(1) $\|T\|\geq 0$

(2) $T=0\Rightarrow\|T\|=0$

$\|T\|=0\Rightarrow Tx=0\ \forall x\neq 0$, i.e., $T=0$

(3) $\|\lambda T\|=\sup\limits_{x\neq 0}\dfrac{\|\lambda Tx\|}{\|x\|}=\sup\limits_{x\neq 0}\dfrac{|\lambda|\cdot\|Tx\|}{\|x\|}=|\lambda|\cdot\|T\|$

(4) $\|T+S\|\leq\|T\|+\|S\|$ for $T,S\in B(X,Y)$.

Thm. X normed space, Y Banach space

$\Rightarrow\left(B(X,Y),\|\cdot\|\right)$ Banach space

Pf.: Let $\{T_n\}\subseteq B(X,Y)$ be Cauchy.

$\forall x\in X$, consider $\{T_nx\}$

$\because\|T_nx-T_mx\|=\|(T_n-T_m)x\|\leq\|T_n-T_m\|\cdot\|x\|\to 0$ as $n,m\to\infty$.

$\therefore\{T_nx\}$ Cauchy in Y

$\Rightarrow T_nx\to y\equiv Tx$ (i.e., pointwise conv.)

(1) T linear transf:

$$T(ax+by)=\lim_n T_n(ax+by)=\lim_n T_n(aT_nx+bT_ny)=a\lim_n T_nx+b\lim_n T_ny=aTx+bTy$$

(2) T bdd:

$\because\{T_n\}$ Cauchy in Y

$\Rightarrow\{\|T_n\|\}$ bdd $\left(\because\|T_n\|\leq\|T_n-T_N\|+\|T_N\|\ \ \forall n\geq N\right)$

Say, $\|T_n\|\leq M\quad\forall$n

$\therefore\underset{\downarrow\ \|Tx\|}{\|T_nx\|}\leq\|T_n\|\cdot\|x\|\leq M\cdot\|x\|\ \ \forall x$

$\Rightarrow\dfrac{\|Tx\|}{\|x\|}\leq M\ \ \forall x\neq 0$

i.e., $\|T\|\leq M$

(3) $T_n \to T$ in $\|\cdot\|$ (i.e., conv. unif. on unit disc)

$\because \forall \varepsilon > 0,\ \exists N \ni m, n \geq N \Rightarrow \|T_m - T_n\| \leq \varepsilon$

$\therefore \|T_m x - T_n x\| \leq \|T_m - T_n\| \cdot \|x\| \leq \varepsilon \cdot \|x\| \quad \forall m, n \geq N$

$\downarrow$

Let $m \to \infty$ $\|Tx - T_n x\|$

$\therefore \|T - T_n\| \leq \varepsilon \quad \forall n \geq N$

i.e., $\|T - T_n\| \to 0$ as $n \to \infty$

Def. X Banach space

$X \times X \to X$ + (1) within: associative law

$(x, y) \mapsto x \cdot y$ (2) with +: distributive law

(3) with $\cdot$: $\lambda(x \cdot y) = (\lambda x) \cdot y = x \cdot (\lambda y)$

(4) with $\|\cdot\|$: $\|x \cdot y\| \leq \|x\| \cdot \|y\|$

Then X Banach algebra

Ex.1. X compact (metric) space,

Then $C(X)$ Banach algebra (with pointwise multiplication)

Pf: $\|fg\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty$

Note: commutative

Ex.2. X Banach space.

Then $B(X)$ Banach algebra (with composition: $S \cdot T = S \circ T$)

Pf: $\|ST\| \leq \|S\| \cdot \|T\| \ \to \ \|S(Tx)\| \leq \|S\| \cdot \|Tx\| \leq \|S\| \cdot \|T\| \cdot \|x\| \quad \forall x$

Note: non-commutative

Note: There's more structure to it: conjugate.

$\Rightarrow C^*$-algebra, von Neumann algebra

Homework:

Sec. 4.4.

Ex. 4.4.6~8

Ex.3. $L^1(\mathbb{R}^n)$ with $(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy$ (convolution) (Ex. 4.4.10)

Then Banach algebra.

Pf: $\|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1$

Note. commutative

Three main results of functional analysis:

(1) Uniform bddness principle
(2) Open mapping thm

} : need completeness.

(3) Hahn-Banach thm: need convexity.

Sec. 4.5. Principle of uniform bddness (Banach-Steinhaus Thm)

Thm. X Banach space, Y normed space.

$\{T_\alpha\}: X \to Y$ bdd operators.

If $\forall x \in X$, $\{\|T_\alpha x\|\}$ bdd, then $\{\|T_\alpha\|\}$ bdd.

Note. $\{\|T_\alpha\|\}$ bdd $\Rightarrow \{\|T_\alpha x\|\}$ bdd $\forall x \in X$

Pf. 1. Check: $\exists\, B(x_0,\varepsilon)$, $\exists\, K > 0 \ni \|T_\alpha x\| \le K \ \forall x \in B(x_0,\varepsilon)\ \forall \alpha$

"Pointwise bdd to local bdd"
(need: completeness)

Assume contrary.

(i) Fix $B(x_0,\varepsilon)$ & $K = 1 \Rightarrow \exists\, x_1 \in B(x_0,\varepsilon), \alpha_1 \ni \|T_{\alpha_1} x_1\| > 1$.

$\because T_{\alpha_1}$ conti.

$\therefore \|T_{\alpha_1} x\| > 1$ on some $\overline{B(x_1,\varepsilon_1)} \subseteq B(x_0,\varepsilon)$ & $\varepsilon_1 < 1$.

(ii) For $B(x_1,\varepsilon_1)$ & $K = 2 \Rightarrow \exists\, x_2 \in B(x_1,\varepsilon_1), \alpha_2 \ni \|T_{\alpha_2} x_2\| > 2$

$\because T_{\alpha_2}$ conti.

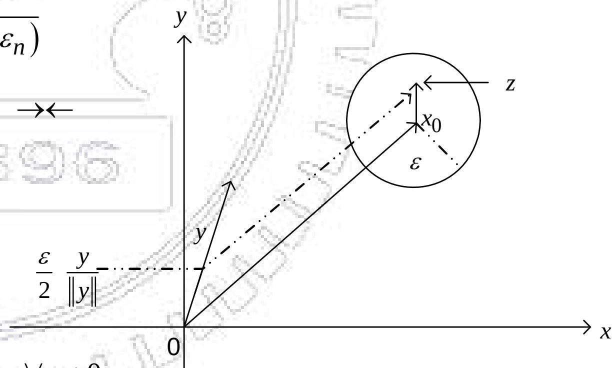
$\therefore \|T_{\alpha_2} x\| > 2$ on some $\overline{B(x_2,\varepsilon_2)} \subseteq B(x_1,\varepsilon_1)$ & $\varepsilon_2 < \frac{1}{2}$

$\therefore \exists\, \underline{\underline{x_n}}, \underline{\underline{\alpha_n}}, \underline{\underline{\varepsilon_n}} < \frac{1}{n} \ni \overline{B(x_n,\varepsilon_n)} \le B(x_{n-1},\varepsilon_{n-1})$ & $\|T_{\alpha_n} x\| > n$ on $\overline{B(x_n,\varepsilon_n)}$.

$\because \overline{B(x_n,\varepsilon_n)} \downarrow$, nonempty, closed in complete X & $\varepsilon_n \to 0$

$\Rightarrow \exists\, z \in \bigcap_n \overline{B(x_n,\varepsilon_n)}$

$\because \|T_{\alpha_n} z\| > n \quad \forall n \ \to\leftarrow$



"local bdd to global bdd"
(need: linearity)

Check: $\|T_\alpha\| \le K \ \forall \alpha$

$\|$

$\sup_{y \ne 0} \frac{\|T_\alpha y\|}{\|y\|}$

Check: $\|T_\alpha y\| \le K.\|y\| \ \forall \alpha, \forall y \ne 0$

Let $z = \frac{\varepsilon/2}{\|y\|} \cdot y + x_0$

Then $z \in B(x_0,\varepsilon)$

$\therefore \|T_\alpha z\| \le K \ \forall \alpha$

$$\Rightarrow \|T_\alpha y\| = \frac{\|y\|}{\frac{\varepsilon}{2}} \|T_\alpha (z - x_0)\| \le \frac{2\|y\|}{\varepsilon}\left(\underset{K}{\underset{\wedge\backslash}{\|T_\alpha z\|}} + \underset{M}{\underset{\wedge\backslash}{\|T_\alpha x_0\|}}\right) \quad \text{(by assumption)}$$

$$\le \|y\| \cdot \frac{2}{\varepsilon}(K + M)$$

Pf. 2. (Ex. 4.5.1)

(Use category thm) (due to S.Saks) in the paper of Banach-Steinhaus. (1927)

"pointwise to local" (need Baine)

- Let $X_n = \left\{ x \in X : \sup_{\alpha} \|T_\alpha x\| \le n\|x\| \right\}$ closed in X
- $\therefore X = \bigcup_n X_n$
- $\because X$ complete metric space
- $\therefore$ Baire category thm $\Rightarrow$ for some n, $X_n \supseteq$ nonempty open set

"local to global" (need linearity) (as in Pf.1)

$\therefore \exists\, B(x_0, r) \subseteq X_n$

$\therefore \forall\, x \in X,\ \|x\| = 1,$

$x_0 + \varepsilon x \in B(x_0, r) \subseteq X_n$ for some $\varepsilon > 0$.

$\therefore \|T_\alpha(x_0 + \varepsilon x)\| \le n\|x_0 + \varepsilon x\| \le n\|x_0\| + n\varepsilon \quad \forall \alpha$

$\vee\!/$

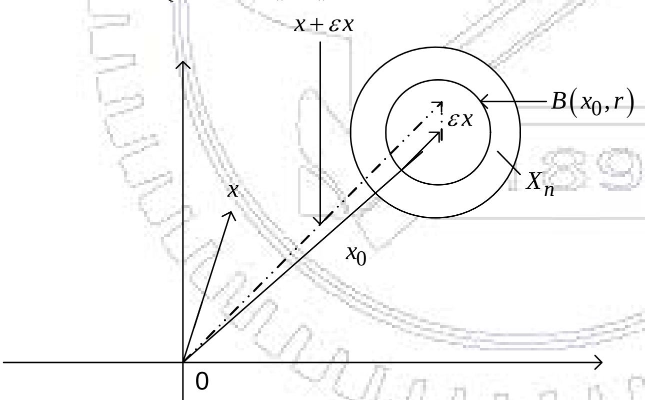
$\varepsilon\|T_\alpha x\| - \|T_\alpha x_0\|$

$\therefore \|T_\alpha x\| \le \frac{1}{\varepsilon}\left(\|T_\alpha x_0\| + n\|x_0\| + n\varepsilon\right)$

$\le \frac{1}{\varepsilon}\left(n\|x_0\| + n\|x_0\| + n\varepsilon\right) \ (\because x_0 \in X_n)\ \forall \alpha$

$\uparrow$ indep. of α & x

$\Rightarrow \|T_\alpha\| \le K \ \forall \alpha$



Note: The same argument applies to "$\forall x \in Z, \{\|T_\alpha x\|\}$ bdd $\Rightarrow \{\|T_\alpha\|\}$ bdd, where Z 2nd category".

Pf. 3. Gliding hump method (Hahn, 1922)

Ref.

1. H.L.Royden, Aspects of constructive analysis, Errett Bishop: Reflections on him and his research, Contemporary Math., Vol. 39, AMS, 1985.

   Thm. X Banach space, Y normed space

   $T_n : X \to Y \ni \|T_n\| > n \cdot 3^n \ \forall n$

   Then $\exists\, x \in X \ \ni \ \|T_n x\| > n$

2. P.R.Halmos, A Hilbert space problem book, Prob. 27
3. J.B.Comway, A course in functional analysis, pp. 98~99