Class 48

X, Y normed spaces

Def. $T_n, T : X \to Y$ bdd operator

$T_n \to T$ in norm if $\|T_n - T\| \to 0$ as $n \to \infty$. (conv. uniformly on unit ball)

$T_n \to T$ strongly if $\|T_n x - Tx\| \to 0$ as $n \to \infty$ $\forall x \in X$ (conv. pointwise)

Note 1. Same as "uniform conv." & "pointwise conv." of functions.

2. $T_n \to T$ in norm $\underset{\not\Leftarrow}{\Rightarrow}$ $T_n \to T$ strongly

Pf: $\forall x \in X$, $\|T_n x - Tx\| \le \|T_n - T\| \cdot \|x\| \to 0$

Ex. $T_n : l^2 \to l^2$

$\ni$ $T_n(x_0, x_1, \ldots) = (x_n, x_{n+1}, \ldots)$ (left shift)

$T = 0$

Then $\|T_n x\|^2 = \sum_{j=n}^{\infty} |x_j|^2 \to 0$ as $n \to \infty$ $\forall x \in l^2$.

$\therefore T_n \to 0$ strongly

But $\|T_n\| = 1$ $\forall n$

$\therefore T_n \not\to 0$ uniformly

3. T_n, T on finite-dim X. Then $T_n \to T$ unif. $\Leftrightarrow$ strongly (need: norms are all equiv.)

Applications of unif. bddness principle:

(1) Thm. X Banach space, Y normed space.

$T_n : X \to Y$ bdd operator

If $\forall x \in X$, $T_n x$ converges, then $\exists$ $T : X \to Y$ bdd operator $\ni$ $T_n \to T$ strongly.

Pf: Let $Tx = \lim_n T_n x$ $\forall x$.

Then (1) T linear:

$$T(ax + by) = \lim_n T_n(ax + by) = \lim_n (aT_n x + bT_n y) = aTx + bTy$$

(2) T bdd:

$\because \{T_n x\}$ bdd $\forall x$

Unif. bddness principle $\Rightarrow \{\|T_n\|\}$ bdd, say, $\|T_n\| \le K$, $\forall n$

$\therefore \underset{\substack{\downarrow \\ \|Tx\|}}{\|T_n x\|} \le \|T_n\| \cdot \|x\| \le K \cdot \|x\| \cdot$ $\forall x$

$\Rightarrow \|T\| \le K$

(3) $T_n \to T$ strongly, by def. of T

(2) $\exists$ conti. func. f on $[0,2\pi]$ $\ni$ $f(0)=f(2\pi)$ & its Fourier series div. at $x=0$

(cf. Ex. 4.5.2~4.5.5) (cf. W.Rudin, Real and complex analysis, p.101)

Def. $f \in L^1[0,2\pi]$

$$S(x)=\sum_{n=-\infty}^{\infty} a_n e^{inx}, a_n=\frac{1}{2\pi}\int_0^{2\pi} f(y)e^{-iny}dy$$

(Fourier series of f)

Note 1. (Kolmogorov, 1926)

$\exists f \in L^1[0,2\pi]$ $\ni$ S_n div. everywhere on $[0,2\pi]$.

2. (L.Carleson, 1966)

$f \in L^2[0,2\pi] \Rightarrow S(x)=f(x)$ a.e.

References:

1. H.L.Royden, Real analysis.
2. B.Gelbaum, Problems in analysis.
3. A.A.Kirillov & A.D.Gvishiani, Theorems and problems in functional analysis.
4. A.E.Taylor & D.C.Lay, Introduction to functional analysis.

Note: $\{a_n\} \in l^q, \{x_n\} \in l^p \left(\frac{1}{p}+\frac{1}{q}=1\right) \Rightarrow \{a_n x_n\} \in l^1$

(3) $\sum_{n=1}^{\infty}|a_n x_n|<\infty \ \forall \{x_n\} \in l^p \ (1<p<\infty) \Rightarrow \{a_n\} \in l^q \left(\frac{1}{p}+\frac{1}{q}=1\right)$

Note:" $\Leftarrow$ " by Hölder's $\le$

(cf. Hellinger-Toeplitz Thm & Halmos, Prob. 29)

Pf.: For each $n \ge 1$, consider

$$T_n\{x_j\}=\sum_{j=1}^{n} a_j x_j : l^p \to \mathrm{R}$$

$$\left|\sum_{j=1}^{n} a_j x_j\right| \le \left(\sum_{j=1}^{n}|a_j|^q\right)^{\frac{1}{q}} \cdot \left(\sum_{j=1}^{n}|x_j|^p\right)^{\frac{1}{p}}$$

by Hölder's $\le$

$$\Rightarrow \|T_n\| \le \left(\sum_{j=1}^{n}|a_j|^q\right)^{\frac{1}{q}}$$

$$\because \forall \{x_j\} \in l^p,\ \left|T_n\{x_j\}\right| \le \sum_{j=1}^{n}|a_j x_j|$$

$$\le \sum_j |a_j x_j| < \infty$$

$\therefore$ unif. bdd principle

$\Rightarrow \|T_n\| \le M \ \ \forall n$

But let $x_j = \begin{cases} \dfrac{|a_j|^{q/p}}{\left(\sum_{j=1}^{n} |a_j|^q\right)^{1/p}} & \text{if } 1 \le j \le n \\ 0 & \text{if } j > n \end{cases}$

Then $\|x\|_p = 1$ & $|T_n x| = \sum_{j=1}^{n} |a_j x_j| = \sum_{j=1}^{n} \dfrac{|a_j|^{1+q/p}}{\left(\sum_{j=1}^{n} |a_j|^q\right)^{1/p}}$

$$= \sum_{j=1}^{n} \frac{|a_j|^q}{\left(\sum_{j=1}^{n} |a_j|^q\right)^{1/p}} = \left(\sum_{j=1}^{n} |a_j|^q\right)^{1/q}$$

$\therefore \{a_n\} \in l^q$