Class49

Note: $T:\underline{\bar{X}}\to\bar{Y}$ Banach spaces. Then either $T\underline{\bar{X}}$ 1st category or $T\underline{\bar{X}}=\bar{Y}$

(cf. p.144, Ex.4.6.5) (Homework)

(Inverse mapping Thm)

$\underline{\bar{X}},\bar{Y}$ Banach spaces

$T:\underline{\bar{X}}\to\bar{Y}$ 1-1, onto, bdd linear tiaraf.

Then $T^{-1}:\bar{Y}\to\underline{\bar{X}}$ onto, bdd linear tiaraf.

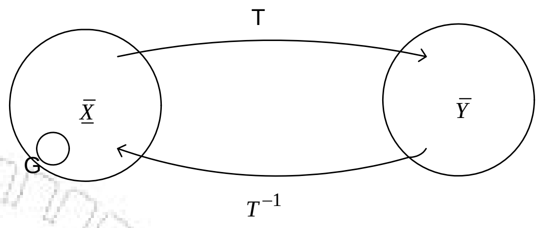
Pf: Check: T^{-1} bdd $\Leftrightarrow T^{-1}$ conti.

Let $G\subseteq\underline{\bar{X}}$ open

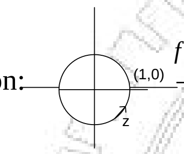
Check: $\left(T^{-1}\right)(G)$ open in $\bar{Y}$

$\Downarrow$
TG

(by open mapping Thm)



Note: In general not true for nonlinear mapping

[Reason: 

$\xrightarrow{f^{-1}} f^{-1}(z)\to 2\pi\neq f^{-1}\left(\underset{\substack{\Downarrow\\0}}{1},0\right)$]

Ex. $f(t)=(\cos t,\sin t):[0,2\pi)\to$ circle. Then f_{1-1}, onto, conti., but f^{-1} not conti at $(1,0)$

Cov. 2. $\left(\underline{\bar{X}},\|\cdot\|_2\right),\left(\underline{\bar{X}},\|\cdot\|_2\right)$ Banach spaces

If $\exists\, K>0 \ni \|x\|_1\le K\cdot\|x\|_2\ \ \forall x\in\underline{\bar{X}}$, then $\|\cdot\|_1\sim\|\cdot\|_2$

Pf.: Let $I:\left(\underline{\bar{X}},\|\cdot\|_2\right)\to\left(\underline{\bar{X}},\|\cdot\|_1\right)$ be the identity mapping.

Then I 1-1, onto, bdd linear transf.

$\Rightarrow I^{-1}:\left(\underline{\bar{X}},\|\cdot\|_1\right)\to\left(\underline{\bar{X}},\|\cdot\|_2\right)$ bdd

$\therefore \|x\|_2\le K_1\cdot\|\cdot\|_1\ \ \forall x$

Closed graph thm.:

$\underline{\bar{X}},\bar{Y}$ Banach spaces

$T:\underline{\bar{X}}\to\bar{Y}$ linear transf.

Assume $G_T=\left\{(x,Tx):x\in\underline{\bar{X}}\right\}\subseteq\underline{\bar{X}}\times\bar{Y}$ (graph of T) is closed

Then T is bdd.

Note: $T \Rightarrow G_T$ closed. (even if T not linear)

Pf: Assume $(x_n, Tx_n) \to (x, y)$

Check: $(x, y) \in G_T$

i.e. $y = Tx$

$\because x_n \to x \;\&\; Tx_n \to y$

$\Downarrow_{Tx_n \to Tx}$

$\to y = Tx$

Pf.: $\because G_T$ closed subspace of $\underline{\overline{X}} \times \overline{Y}$

$$\begin{pmatrix} \because (x, Tx) + (y, Ty) = (x+y, T(x+y)) \in G_T \\ \lambda (x, Tx) = (\lambda x, \lambda Tx) \in G_T \end{pmatrix}$$

Assume $(\underline{\overline{X}} \times \overline{Y}, \| \ \|), \|x, y\| = \|x\| + \|y\|$, Banach space

$\Rightarrow G_T$ Banach space

Let $J : G_T \to \underline{\overline{X}}$ be $\ni J(x, Tx) = x$

Then $J, 1-1,$ onto, bdd linear transf. $(\because \|x\| \le \|(x, Tx)\| = \|x\| + \|Tx\|)$

Cov. 1 $\Rightarrow J^{-1} 1-1,$ onto, bdd linear transf. $: \underline{\overline{X}} \to G_T$

$\therefore \exists K > 1 \ni \|J^{-1}x\| \le K \cdot \|x\| \quad \forall x \in \underline{\overline{X}}$

$\Downarrow$

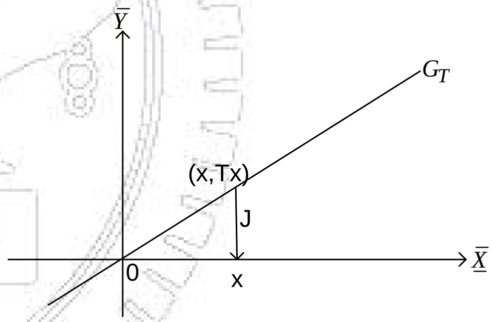
$\|(x, Tx)\|$

$\Downarrow$

$\|x\| + \|Tx\|$

$\Rightarrow \|Tx\| \le (K-1) \quad \forall x \in \underline{\overline{X}}$

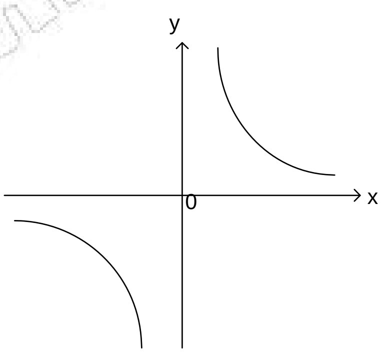
$\Rightarrow T$ bdd



closed graph thm:

Note 1. Not true if T not linear.

Ex. $T : \mathbb{R} \to \mathbb{R} \ni T(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$



Then G_T closed, but T not conti. at 0

Note 1. Open massive thm $\Leftrightarrow$ closed graph thm

Note: Closed graph theorem $\Rightarrow$ Inverse mapping thm

Pf: $\underline{\overline{X}}, \overline{Y}$ Banach spaces

Let $T.\underline{\overline{X}} \to \overline{Y}$ 1-1, onto, bdd, linear transf.

Check. $T^{-1} : \overline{Y} \to \underline{\overline{X}}$ bdd

Check. $G\left(T^{-1}\right)=\left\{\left(y,T^{-1}y\right) y\in\bar{Y}\right\}$ closed

Say, $\left(y_n,T^{-1}y_n\right)\to(y,x)$

$\therefore y_n\to y \,\&\, T^{-1}y_n\to x$

$\Rightarrow y_n\to Tx$

$\therefore y=Tx$

$\therefore x=T^{-1}y$

inverse mapping thm

$(y,x)=\left(y,T^{-1}y\right)\in G\left(T^{-1}\right)$

Note: $\Rightarrow$ Open mapping thm.

Let $T:\underline{\bar{X}}\to\bar{Y}$ onto, bdd

Pf.: Check: T open

Let $G\subseteq\underline{\bar{X}}$ open

Check: TG open

Consider $\tilde{T}: \underline{\bar{X}}/\ker T \xrightarrow{\tilde{x}\to Tx} \bar{Y}$ 1-1, onto, bdd linear transf. $\Rightarrow \tilde{T}$ open

Let $\pi:\underline{\bar{X}}/\ker T:\pi(x)=\tilde{x}$

Then π open $\Rightarrow T=\tilde{T}^{0\pi}$ is open

$\because \underline{\bar{X}}/\ker T$ has norm $\|\tilde{x}\|=\inf\{\|y\|: y-x\in\ker T$

Then Banach space

Check: $\pi\left(\left\{x:\|x-x_0\|<\gamma\right\}\right)$ is open

$=\left\{\tilde{x}:\|x-x_0\|<\gamma\right\}$

Conclusion: "open mapping thm", "inverse mapping thm" & "closed graph thm" are all equiv.

Note: $T:\underline{\bar{X}}\to\bar{Y}$ linear, $\underline{\bar{X}},\bar{Y}$ Banach spaces

T bdd $\Leftrightarrow$ "$x_n\to x\Rightarrow Tx_n\to Tx$".

Closed graph thm $\Rightarrow$ Check:"$x_n\to x$ & $Tx_n\to y\Rightarrow Tx=y$"