Class5

Ex. (Ex.1.4.1) Let $K=\{\phi, X, \text{singletons}\}$;
$\lambda(\phi)=0$, $\lambda(X)=\#X$, $\lambda(E)=1$ if E singleton.
Then $u^*(A)=\#A$; outer measure
$\boldsymbol{a}=\wp(X)$; σ-algebra
$\therefore u^*$ measure on $\wp(X)$ (counting measure); measure.

Homework: Ex.1.4.4, 1.4.5

Sec. 1.5.

$(X, \boldsymbol{a}, u)$ measure space.

Def. u is complete if $E\in\boldsymbol{a},\ u(E)=0,\ N\subseteq E \Rightarrow N\in\boldsymbol{a}$.

Note: u constructed from outer measure u^* is complete

Pf. Check: $u^*(A)\geq u^*(A\setminus N)+u^*(A\cap N)\quad \forall A$

$\wedge\setminus \leftarrow\because u^*$ monotone

$u^*(E)$

$\|\quad \leftarrow\because u^*|\boldsymbol{a}=u$

$u(E)=0$

	X set	
	λ,	$K\subseteq\wp(X)$ sequential convering class
outer measure	u^*	$\wp(X)$
		$\cup/$
measure	u	$\boldsymbol{a}$
complete measure	$\overline{u}$	$\overline{\boldsymbol{a}}$

Ex. $\boldsymbol{a}=\{\phi, X\}$: σ-algebra
$u(\phi)=u(X)=0$
Then u measure, not complete, if #A ≥ 2
Thm $\Rightarrow \overline{\boldsymbol{a}}=\wp(X)$
$\overline{u}\equiv 0$.

Thm. u measure on $\boldsymbol{a}$

(1) Let $\overline{\boldsymbol{a}}=\{E\cup N:\ E\in\boldsymbol{a},\ N\subseteq A,\ \text{where } A\in\boldsymbol{a}\ \&\ u(A)=0\}$ ($\Rightarrow \overline{\boldsymbol{a}}\supseteq\boldsymbol{a}$)

Then $\overline{\boldsymbol{a}}$ σ-algebra

(2) Let $\overline{u}(E\cup N)=u(E)$ for $E\cup N\in\overline{\boldsymbol{a}}$.

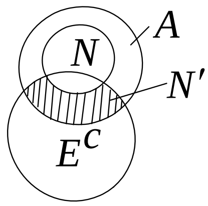
Then $\overline{u}$ complete measure on $\boldsymbol{a}$. ($\Rightarrow \overline{u}|\boldsymbol{a}=u$)

Note: Other construction of $\bar{\boldsymbol{a}}$: $E \setminus N$ (Ex.1.5.2) or $E\Delta N$

Pf. (1) $\because \phi=\phi\cup\phi \in \bar{\boldsymbol{a}}$

Check: $E\cup N \in \bar{\boldsymbol{a}} \Rightarrow (E\cup N)^c \in \bar{\boldsymbol{a}}$.

Say, $E \in \boldsymbol{a}$, $N \subseteq A \in \boldsymbol{a}$, $u(A)=0$.



$$\because (E\cup N)^c = E^c \cap N^c = E^c \setminus N = (E^c \setminus A)\cup N' \in \bar{\boldsymbol{a}}.$$

$$\underset{\boldsymbol{a}}{\Cap}\qquad \underset{A}{\cap|}$$

Check: $E_n \cup N_n \in \bar{\boldsymbol{a}} \Rightarrow \bigcup_n (E_n\cup N_n) \in \bar{\boldsymbol{a}}$

$$\| $$

$$(\bigcup_n E_n)\cup(\bigcup_n N_n)$$

$$\underset{\boldsymbol{a}}{\Cap}\qquad \bigcup_n A_n \in \boldsymbol{a} \ \& \ u(\bigcup_n A_n) \le \sum_n u(A_n) = 0$$

(Say, $E_n \in \boldsymbol{a}$, $N_n \subseteq A_n \in \boldsymbol{a}$, $u(A_n)=0$)

$\Rightarrow \bar{\boldsymbol{a}}$ is σ-algebra

(2) Check: $\bar{u}$ well-defined.

Let: $E_1\cup N_1 = E_2\cup N_2$, where $E_1, E_2 \in \boldsymbol{a}$, $N_1 \subseteq A_1$, $N_2 \subseteq A_2$, $A_1, A_2 \in \boldsymbol{a}$, $u(A_1)=u(A_2)=0$.

Check: $u(E_1)=u(E_2)$

$\because E_1 \subseteq E_2 \cup N_2 \subseteq E_2 \cup A_2$

$\Rightarrow u(E_1) \le u(E_2) + u(A_2)$

$$\|$$

$$0$$

Similarly, $u(E_2) \le u(E_1)$.

Note:(1) $\bar{u}(\phi)=u(\phi)=0$

(2) $\bar{u}(\bigcup_n (E_n\cup N_n)) = \bar{u}((\bigcup_n E_n)\cup(\bigcup_n N_n)) = u(\bigcup_n E_n) = \sum_n u(E_n) = \sum_n \bar{u}(E_n\cup N_n)$

for disjoint $\{E_n \cup N_n\}$

$\Rightarrow \bar{u}$ is measure

(3) $B \subseteq E\cup N \in \bar{\boldsymbol{a}}$, $\bar{u}\,(E\cup N)=0 \Rightarrow u(E)=0$

Say, $N \subseteq A$, $A \in \boldsymbol{a}$ & $u(A)=0$

Then $B \subseteq E\cup A$, where $u(E\cup A)=0$, $E\cup A \in \boldsymbol{a}$

$\Rightarrow B \in \bar{\boldsymbol{a}}$ (by def. of $\bar{\boldsymbol{a}}$).

$\Rightarrow \bar{u}$ is a complete measure.

Homework: Ex.1.5.1, 1.5.2

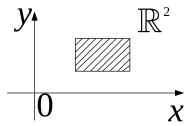
Sec. 1.6. Lebesgue measure

(1) Lebesgue measure on $\mathbb{R}^n$:

$\mathbb{R}^n$

$K=\{\text{bdd open intervals in } \mathbb{R}^n\}\cup\{\phi\}$ sequential covering class.

$\lambda(\text{bdd open interval}) = \text{its volume}$.



$\Rightarrow u^*$ Lebesgue outer measure

u Lebesgue measure (complete) on $\boldsymbol{a}$

$\to$ Lebesgue measurable subets of $\mathbb{R}^n$

may prove intervals are Lebesgue measurable

$\to$ need topological consideration

(2) Lebesgue-Stieltjes measure on $\mathbb{R}$:

Let $f:\mathbb{R}\to\mathbb{R}$ be $\uparrow$ & right conti.

$\mathbb{R}$

$K=\{(a,b]: a<b\in\mathbb{R}\}\cup\{\phi\}$ sequential covering class.

$\lambda((a,b]) = f(b)-f(a)$.

$\Rightarrow u^*$ Lebesgue-Stieltjes outer measure

u_f Lebesgue-Stieltjes outer measure

(complete) on $\boldsymbol{a}_f$ & $u_f((a,b]) = f(b)-f(a)$ (Ex. 1.9.13)

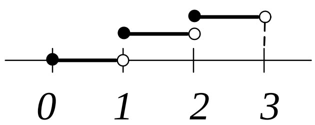
Note: If $K=\{(a,b)\}$, $\lambda((a,b)) = f(b)-f(a)$ Then $u_f((a,b)) \le f(b)-f(a)$. (Ex. 1.9.15)

Ex1. $f(x)=x$

Then u_f Lebesgue measure.

Ex2. $f(x)=[x]$ $\uparrow$ & right conti.

Then u_f is defined on $\boldsymbol{a}_f$ $\ni$ $u_f(E)$=no. of integers in E.



Reason: $u_f((\frac{1}{2},3])=[3]-\left[\frac{1}{2}\right]=3-0=3$

Note: $f \rightleftarrows u_f$

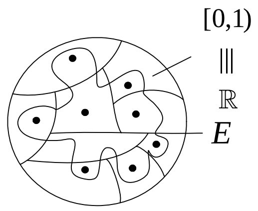
For certain u, let $f(x)=u((-\infty,x])$ (distribution function)

Then f $\uparrow$ & right conti.

Homework: Ex.1.6.3

Thm. $E \subseteq \mathbb{R} \ni E$ not Lebesgue measurable.

i.e. In (1) above, $\boldsymbol{a} \subsetneq \wp(\mathbb{R})$



$x, y \in [0,1)$

Def. $x \equiv y$ if $x - y$ rational

Then "$\equiv$" equiv. relation.

$\therefore$ Axiom of choice

$\Rightarrow F \equiv$ set formed by taking 1 element from each equiv. class

Check: F not Lebesgue measurable.

Assume F Lebesgue measurable

$\Rightarrow \frac{1}{k} + F$ Lebesgue measurable & $u(\frac{1}{k} + F) = u(F) \quad \forall k \geq 1$

Note: $\left\{\frac{1}{k} + F\right\}$ mutually disjoint

Let $x \in (\frac{1}{k} + F) \cap (\frac{1}{l} + F)$ for $k \neq l$

$\therefore x = \frac{1}{k} + \eta_1 = \frac{1}{l} + \eta_2$ for some $\eta_1, \eta_2 \in F$

$\Rightarrow \eta_1 - \eta_2 =$ rational $\Rightarrow \eta_1 \equiv \eta_2 \Rightarrow \eta_1 = \eta_2 \Rightarrow k = l \quad \rightarrow\leftarrow$

$\therefore nu(F) = \sum_{k=1}^{n} u(\frac{1}{k} + F) = u(\bigcup_{k=1}^{n} (\frac{1}{k} + F)) \leq u([0,1)) = 1 \quad \forall n \geq 1$

$\Rightarrow u(F) = 0$

Consider $r + F$ for rational r

$\Rightarrow r + F$ Lebesgue measurable & $u(r + F) = u(F) = 0$

Also, $(r + F) \cap (s + F) = \phi$ for $r \neq s$ rational

$\bigcup_{r} (r + F) = [0,1) \Rightarrow u([0,1)) = 0 \quad \rightarrow\leftarrow$

Reason: "$\subseteq$": ✓

"$\supseteq$": Let $x \in [0,1)$

$\therefore x \in$ some equiv. class

$\Rightarrow x \equiv \eta$ for some $\eta \in F$

$\therefore x = \eta + r$ for some rational r

$\therefore x \in$ LHS

u Lebesgue measure on $\mathbb{R}$:

(1) $u(\{a\})=0$.   (Ex.1.6.1)

Easy: by (3)

(2) $u(\{a_1,a_2,...\})=0$ (Ex.1.6.2)

$\because$ countable subadditivity

(3) $u([a,b])=b-a$ (Ex.1.6.3)

Need: Heine-Borel thm.

(4) $u((a,b))=u((a,b])=u([a,b))=b-a$ (Ex.1.6.4).

Easy

(5) E Lebesgue measurable, $a\in\mathbb{R}\Rightarrow a+E$ Lebesgue measurable & $u(a+E)=u(E)$.