Class50

Applications

(2) For $f\in L'[0,2\pi]$,let $a_n=\frac{1}{2\pi}\int_0^{2\pi}f(y)e^{-iny}dy.n=...-2,-1,0,1,2...$ (Fourier coeffi.)

Mercel's thm:

$$\lim_{n\to\pm\infty}a_n=0\left[\begin{array}{l}\text{Pf: Parseval's equalitz}\Rightarrow\text{time for }f\in L^2[0,2\pi]\\ \text{If }f\in L',\text{ then }\exists f_k\in L^2\ni f_k\to f\text{ in }\|.\|_1\Rightarrow a_n(f_k)\to a_n(f)\text{ unif in }n\text{ as }k\to\infty\end{array}\right\}$$

Question: $\{a_n\}\ni\lim_{n\to\pm\infty}a_n=0\overset{?}{\Rightarrow}\exists f\in L'[0,2\pi]\ni\{a_n\}$ Fourier coeffi's of f

Ans. No. (by inverse mapping thm)

Reason: Let $T:L'[0,2\pi]\to C_0=\left\{\{a_n\}:\lim_{n\to\pm\infty}a_n=0\right\}$ (with $\|.\|_\infty$)

Then T is 1-1, bdd, linear transf.,

$$\because|a_n|\le\frac{1}{2\pi}\int_0^{2\pi}|f(y)|dy$$
$$\le\|f\|_1$$
$$\Rightarrow\|\{a_n\}\|_\infty\le\|f\|_1$$

(cf. W.Rudin, Real and complex analysis, pp.103-104)

If, T onto, then $L'[0,2\pi]\cong C_0$

$$\Rightarrow L'[0,2\pi]^*\cong C_0{}^*$$

$$\overset{\cong}{L^\infty[0,2\pi]}\qquad\overset{\cong}{e'}$$

nonseparable separable $\to\leftarrow$

Let $x\in\underline{X}$

Check: T conti. at $x\in\underline{X}$

Assume $x_n\to x$

Check: $Tx_n\to Tx$

$\because x_n\to x\Rightarrow x_n-x+z\to z$

$\Rightarrow T(x_n-x+z\to z)\to Tz$

$\Downarrow$

$Tx_n-Tx+Tz$

$\Rightarrow Tx_n\to Tx$ as needed.

(1) $A=\begin{bmatrix} a_{11} & a_{12} \cdots \\ a_{21} & a_{22} \ldots \\ \vdots & \vdots \end{bmatrix}: l^2 \to l^2 \Rightarrow A$ bdd

Pf.: (Idea: by closed graph thm).

Let $x_n \to x$ & $Ax_n \to y$ in $\|\cdot\|_2$

$$\therefore \langle Ax_n, z\rangle = \left\langle x_n, \overset{*}{A}z \right\rangle$$

$$\downarrow \qquad\qquad \downarrow$$

$$\langle y, z\rangle \qquad \left\langle x, \overset{*}{A}z \right\rangle$$

$$\backslash\backslash \qquad\qquad \|$$

$$\langle Ax, z\rangle \qquad \forall z \in l^2$$

$\Rightarrow y = Ax$

$\therefore$ G_A is closed

$\therefore$ closed graph thm $\Rightarrow A$ bdd

(cf. J.B Conway, A course in functional analysis, 2nd ed., p.93, Ex.7)

Note 1. Ex.4.6.6 not correct

$\underline{\overline{X}}, \overline{Y}$ Banach spaces

$T: \underline{\overline{X}} \to \overline{Y}$ linear transf, $\ker T = \{x \in \underline{\overline{X}} : Tx = 0\} \subseteq \underline{\overline{X}}$

Then T bdd $\underset{\not\Leftarrow}{\Rightarrow}$ $\ker T$ closed in $\underline{\overline{X}}$

Pf.: $\because \ker T = T^{-1}\{0\}$ closed

($\because$ In metric space, singleton closed).

Ex. Let $\underline{X} = C_0 = \{(x_1, x_2, ...) : x_n \in \square, x_n \to 0\}$ (cf.B.Gelbaum, Problems in analysis, Problem 376)

$\overline{Y} = l^1$

Then $(C_0, \|\cdot\|_\infty), (l^1, \|\cdot\|_1)$ Banach spaces.

Check: dim $C_0 = \lim l^1$

$\because \# C_0 = \# l^1 = \aleph_1 \quad (\square \subseteq l^1, C_0 \subseteq \square \times \square \times ......)$

& $\lim C_0, \lim l^1 = \infty$ ($\because \exists$ infinitely many indep. vectors)

> Note 1. $\underline{X}$ Banach space
>
> $\Rightarrow \lim \underline{X} < \infty$ or $\lim \underline{X} \geq \aleph_1$
>
> i.e. No Banach space has a countable Hamel basis (Ex.4.8.7)

$\Rightarrow$ dim $C_0 = \mathrm{dim}\, l^1 = \aleph_1$(Note: $\{x_n = (0, ..., 0, 1, 0, ...)\}$ not a Hamel basis)

n-th

Let $\{x_\alpha\}, \{y_\alpha\}$ Hamel basis for C_0, l^1, resp.

$\therefore \forall x \in C_0$, $x = \sum_\alpha \lambda_\alpha x_\alpha$, where $\lambda_\alpha = 0$ except finitely many α's.

Let $Tx = \sum_\alpha \lambda_\alpha y_\alpha$

Then T is 1-1, onto, linear transf.

> Note 2. $\underline{X}, \overline{Y}$ vector spaces (Ex.4.8.3)
>
> Then $\underline{X}$ isomorphic to $\overline{Y}$ iff dim $\underline{X} = \mathrm{dim}\, \overline{Y}$
>
> (only algebraically)

Note: 3 levels of isomorphism between normed spaces:

(1) isomorphism (algebraically)

(2) homeo. isom. (top+alg.)

(3) isome. isom. (norm+alg.)

$\therefore \ker T = \{0\}$ closed

But T not bdd.

Reason: If T bdd, then, by (1), $C_0 \cong l^1$

(topo & algebraically)

$$\Rightarrow C_0{}^* \cong l^{1*}$$

$$\begin{array}{cc} \cong & \cong \\ l^1 & l^\infty \\ \text{separable} & \text{not separable} \end{array} \to\leftarrow$$

Homework:

Sec.4.6, Ex.2,3,5,6 (modified)

$\underline{\bar{X}}, \bar{Y}$ Banach spaces & $T : \underline{\bar{X}} \to \bar{Y}$ linear

Note 2. T bdd $\Leftrightarrow \ker T$ closed in $\underline{\bar{X}}$ if $\lim \bar{Y} < \infty$

$\| \Leftarrow \|$:

Pf.: Consider $\tilde{T} : \underline{\bar{X}} / \ker T \to \bar{Y}$ 1-1 on Banach spaces (Here "$\ker T$ closed" needed)

$\Rightarrow \lim \underline{\bar{X}} / \ker T \le \lim \bar{Y} < \infty$

$\Rightarrow \tilde{T}$ bdd $\Rightarrow T = \tilde{T}^{0\pi}$ bdd.

Sec.4.8. Hahn-Banach Thm

$\underline{\bar{X}}$ normed space over $F = \mathrm{R}, \mathrm{C}$

Ex. $\underline{\bar{X}} < \mathrm{R}^n$ over R

$$f \in \underline{\bar{X}}^* :\leftrightarrow [y_1, \ldots, y_n] : f(x) = [y_1, \ldots, y_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_i y_i x_i$$

↑

Def. $\underline{\bar{X}}^* = \{ f : \underline{\bar{X}} \to F : f \text{ bdd, linear} \}$ (dual of $\underline{\bar{X}}$)

(linear functional)

↓

Motivation:

In Banach space, $f \in \underline{\bar{X}}^*, x \in \underline{\bar{X}}$, $f(x)$ to replace inner product in Hilbert space.

Note: $\underline{\bar{X}}^*$ Banach space (p.136, Thm.4.4.4)

Hahn-Banach Thm says "$\underline{\bar{X}}^*$ is rich"

Note: Nothing to do with "completeness"; unlike uniform bddness principle & open mapping thm)

$\underline{\bar{X}}$ vector space over R

$P : \underline{\bar{X}} \to \mathrm{R} \ni p(x+y) \le p(x) + p(y)$ (p acts as noun)

$p(\lambda x) = \lambda p(x) \ \forall \lambda \ge 0$

$\bar{Y} \subseteq \underline{\bar{X}}$ subspace

$f : \bar{Y} \to \mathrm{R}$, linear & $f(x) \le p(x) \ \forall x \in \bar{Y}$.

Then f can be extended to $F : \underline{\bar{X}} \to \mathrm{R}$ linear & $F(x) \le p(x) \ \forall x \in \underline{\bar{X}}$

Pf.

Zorn's Lemma:

- Let $K=\left\{\left(\overline{Y}_\alpha, g_\alpha\right):\ \overline{Y}\subseteq \overline{Y}_\alpha \subseteq \underline{X},\ g_\alpha:\overline{Y}_\alpha\to\mathbb{R},\ \text{linear, extend } f\ \&\ g_\alpha(x)\le p(x)\ \forall x\in\overline{Y}_\alpha\right\}$ ($\overline{Y}_\alpha$: subspace)
- Define $\left(\overline{Y}_\alpha, g_\alpha\right)\le\left(\overline{Y}_\beta, g_\beta\right)$ if $\overline{Y}_\alpha\subseteq\overline{Y}_\beta$ & $g_\alpha=g_\beta$ on $\overline{Y}_\alpha$
- Then $(K,\le)$ partially ordered (reflexive, antisymmetric & transitive)
- Also, if $\left\{\left(\overline{Y}_\alpha, g_\alpha\right)\right\}$ totally ordered, let $\overline{Y}'=\bigcup_\alpha \overline{Y}_\alpha$ & $g'(x)=g_\alpha(x)$ if $x\in\overline{Y}_\alpha\subseteq\overline{Y}'$.
  - Then $\left(\overline{Y}',g'\right)\in K$ & $\left(\overline{Y}_\alpha, g_\alpha\right)\le\left(\overline{Y}',g'\right)\ \forall\alpha$.
  - i.e. any totally ordered $\left\{\left(\overline{Y}_\alpha, g_\alpha\right)\right\}$ has an upper bd (in K)
- $\therefore$ Zorn's Lma $\Rightarrow\exists$ max. element $\left(\overline{Y}_0, g_0\right)\in K$.
  - Check: $\overline{Y}_0=\underline{X}$.

Assume $\overline{Y}_0 \underset{\neq}{\subset} \underline{X}$

Let $y_1\in\underline{X}$, but $y_1\notin\overline{Y}_0$

Let $\overline{Y}_1=\left\{y+\lambda y_1: y\in\overline{Y}_0, \lambda\in\mathrm{R}\right\}$: subspace & $\underset{\neq}{\supset}\overline{Y}_0$

Define $g_1:\overline{Y}_1\to\mathbb{R}\ni g_1(y+\lambda y_1)=g_0(y)+\lambda C.$ for some $C\in\mathrm{R}$

Note 1. g_1 well-defined:

Reason: $y+\lambda y_1=y'+\lambda' y_1$

$$\Rightarrow \underbrace{y-y'}_{\in\,\overline{Y}_0}=(\lambda'-\lambda)\underbrace{y_1}_{\notin\,\overline{Y}_0}$$

$$\Rightarrow \lambda'=\lambda\ \ \&\ \ y=y'$$

Note 2. g_1 linear:

Note 3. g_1 extended g_0 (when $\lambda=0$)

Need: $c\ni g_0(y)+\lambda c\le p(y+\lambda y_1)\ \forall y\in\overline{Y}_0$ & $\lambda\in\mathbb{R}$

one-dim extension due to Helly (1912):

Then $\left(\overline{Y}_0, g_0\right) \le \left(\overline{Y}_1, g_1\right)$ & $\overline{Y}_0 \ne \overline{Y}_1$ & $\left(\overline{Y}_1, g_1\right) \in K$.

$\rightarrow\leftarrow$ max. of $\left(\overline{Y}_0, g_0\right)$

Need $c \ni \lambda c \le p\left(y + \lambda y_1\right) - g_0\left(y\right)\ \forall \lambda \in \mathbb{R},\ y \in \overline{Y}_0$

$$\Leftrightarrow (1)\ c \le p\Big(\underbrace{\boxed{\tfrac{y}{\lambda}}}_{z} + y_1\Big) - g_0\Big(\underbrace{\boxed{\tfrac{y}{\lambda}}}_{z}\Big)\quad \forall \lambda > 0, \forall y \in \overline{Y}_0.$$

$$\&\ (2)\ c \ge -p\Big(-\underbrace{\boxed{\tfrac{y}{\lambda}}}_{x} - y_1\Big) - g_0\Big(\underbrace{\boxed{\tfrac{y}{\lambda}}}_{x}\Big)\quad \forall \lambda < 0, \forall y \in \overline{Y}_0.$$

(Note: for $\lambda = 0, g_0\left(y\right) \le p\left(y\right)\ \forall y \in \overline{Y}_0$ holds)

Check: $\Leftrightarrow \sup\limits_{x \in \overline{Y}_0}\left\{-p\left(-x - y_1\right) - g_0\left(x\right)\right\} \le C \le \inf\limits_{z \in \overline{Y}_0}\left\{p\left(z + y_1\right) - g_0\left(z\right)\right\}$

$\forall x, z \in \overline{Y}_0,\ -p\left(-x - y_1\right) - g_0\left(x\right) \le p\left(z + y_1\right) - g_0\left(z\right)$

$$\Updownarrow$$

$$\underbrace{g_0\left(z\right) - g_0\left(x\right)}_{\parallel\ g_0(z-x)} \le p\left(z + y_1\right) + p\left(-x - y_1\right)$$

$\because g_0\left(z - x\right) \le p\left(z - x\right) \le p\left(z + y_1\right) + p\left(-x - y_1\right)\quad \therefore$ OK

Note: <u>Helly intersection Thm</u> in convex analysis: $\left\{a_\lambda, b_\lambda\right\}$ in $\mathbb{R}$ & any two have nonempty intersection

$$\Rightarrow \bigcap_{\lambda}\left[a_\lambda, b_\lambda\right] \ne \varnothing$$

More generally, $\left\{A_\lambda\right\}$ in R^n compact, convex, any $n+1$ have nonempty intersection

$\Rightarrow \bigcap\limits_{\lambda} A_\lambda \ne \varnothing$ In this case, $a_\lambda = -p\left(-x - y_1\right) - g_0\left(x\right)$

$b_\lambda = p\left(x + y_1\right) - g_0\left(x\right)$