Class 52

Let $z^*: Y_1 \to F \ \ni\ z^*\left(y+\lambda x_0\right)=\lambda$

Then linear functional, $z^*\left(y\right)=0\ \forall y\in Y\ \ \&\ \ z^*\left(x_0\right)=1$

Check: $\left\|z^*\right\|=\dfrac{1}{d}$, Check: $\left\|z^*\right\|\le\dfrac{1}{d}$

(i) Check: $\underset{\overset{\|}{|\lambda|}}{\left|z^*\left(y+\lambda x_0\right)\right|}\le\dfrac{1}{d}\cdot\left\|y+\lambda x_0\right\|$

May assume $\lambda\ne 0$ $\Updownarrow$

$$d\le\frac{1}{|\lambda|}\left\|y+\lambda x_0\right\|=\left\|\frac{y}{\lambda}+x_0\right\|$$

$$\Rightarrow\left\|z^*\right\|\le\frac{1}{d}$$

(ii) Let $\left\{y_n\right\}\subseteq Y\ \ni\ \left\|x_0-y_n\right\|\to d\cdot$

$\because \underset{\overset{\|}{1}}{z^*\left(x_0-y_n\right)}\le\left\|z^*\right\|\cdot\left\|x_0-y_n\right\|\to\left\|z^*\right\|d$

$\Rightarrow\left\|z^*\right\|\ge\dfrac{1}{d}$

$\therefore$ Hahn-Banach Thm $\Rightarrow$ extend z^* to $x^*\in X^*$

Duality:

$x^*\ne 0\Leftrightarrow\exists\, x\in X\ni x^*\left(x\right)\ne 0$

$x^*\ne 0\Leftrightarrow\exists\, x^*\in X^*\ni x^*\left(x\right)\ne 0$

(2) X normed space

Let $x\ne 0$ in X

Then $\exists\, x^*\in X^*\ \ni\ x^*\left(x\right)=\|x\|\ \&\ \left\|x^*\right\|=1.$

Note: X Hilbert space $\Rightarrow$ consider $\dfrac{x}{\|x\|}$ & $x^*(y)=\left\langle y,\dfrac{x}{\|x\|}\right\rangle\ \forall y\in X$

Pf: Let $Y=\left\{0\right\}$ in (1)

Then $\exists\, z^*\in X^*\ \ni\ z^*\left(x\right)=1\ \&\ \left\|z^*\right\|=\dfrac{1}{d},\ d=\|x\|$

Let $x^*=\|x\|\cdot z^*$

(3) (D.Lee & P.Y.Wu)

X Banach space

T invertible operator on X

Then $\inf\left\{\|T-S\| : S \text{ noninvertible on } X\right\} = \dfrac{1}{\|T^{-1}\|}$

Pf: Let $\overset{\|}{\alpha}$

(i) $"\geq": \|T-S\| < \dfrac{1}{\|T^{-1}\|} \Rightarrow S$ invertible (Ex.4.6.3)

$\therefore S$ noninvertible $\Rightarrow \|T-S\| \geq \dfrac{1}{\|T^{-1}\|}$

$\therefore \alpha \geq \dfrac{1}{\|T^{-1}\|}$

(Idea: Find noninv. $S_n \ni \|T-S_n\| \to \dfrac{1}{\|T^{-1}\|}$)

(ii) $"\leq": \because \|T^{-1}\| = \sup\limits_{x\neq 0} \dfrac{\|T^{-1}x\|}{\|x\|}$

$$\Rightarrow \therefore \frac{1}{\|T^{-1}\|} = \inf_{x\neq 0} \frac{\|x\|}{\|T^{-1}x\|} = \inf_{y\neq 0} \frac{\|Ty\|}{\|y\|} = \inf_{\|y\|=1} \|Ty\|.$$

$\therefore \exists\, y_n \in X \ni \|y_n\| = 1 \ \& \ \|Ty_n\| \to \dfrac{1}{\|T^{-1}\|}$

(2) $\Rightarrow \exists\, x_n^* \in X^* \ni x_n^*(y_n) = \|y_n\| = 1 \ \& \ \|x_n^*\| = 1.$

Let $S_n x = Tx - x_n^*(x) Ty_n \quad \forall x \in X$

Then S_n bdd, linear

$\because S_n y_n = Ty_n - \underset{\underset{1}{\|}}{x_n^*(y_n)} Ty_n = 0 \ \& \ y_n \neq 0$

$\Rightarrow S_n$ noninvertible

$\alpha \leq \|T-S_n\| \leq \|x_n^*\| \cdot \|Ty_n\| = \|Ty_n\| \to \dfrac{1}{\|T^{-1}\|}$

$\therefore \alpha \leq \dfrac{1}{\|T^{-1}\|}$

$\Rightarrow \alpha = \dfrac{1}{\|T^{-1}\|}$

(4) X normed space

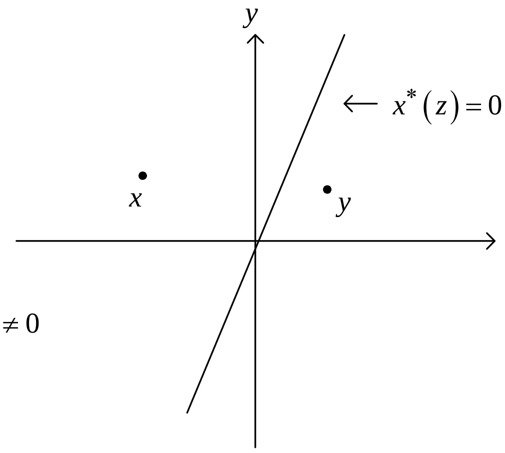
$x \neq y \in X$

Then $\exists\, x^* \in X^* \ni x^*(x) \neq x^*(y)$

Note: X Hilbert space $\Rightarrow \dfrac{x-y}{\|x-y\|}$

Pf: $\because x-y \neq 0$

$$(2) \Rightarrow \exists\, x^* \in X^* \ni \|x^*\| = 1 \;\&\; \underset{\underset{x^*(x)-x^*(y)}{\parallel}}{x^*(x-y)} = \|x-y\| \neq 0$$



(5) $x \in X$ normed space

$$\left.\begin{array}{l} \text{Then } \|x\| = \sup\limits_{x^* \neq 0} \dfrac{|x^*(x)|}{\|x^*\|} = \sup\limits_{\|x^*\|=1} |x^*(x)| \\[2ex] \text{Note: } x^* \in X^* \Rightarrow \|x^*\| = \sup\limits_{x \neq 0} \dfrac{|x^*(x)|}{\|x\|} = \sup\limits_{\|x\|=1} |x^*(x)| \end{array}\right\} \text{duality}$$

Pf: "$\geq$": $\because |x^*(x)| \leq \|x^*\| \cdot \|x\|$

$$\Rightarrow \sup_{x^* \neq 0} \frac{|x^*(x)|}{\|x^*\|} \leq \|x\|$$

"$\leq$": Conversely, for $x \neq 0$, $\exists\, x_0^* \in X^* \ni x_0^*(x) = \|x\|$ & $\|x_0^*\| = 1$ (by (2))

$$\Rightarrow \sup_{x^* \neq 0} \frac{|x^*(x)|}{\|x^*\|} \geq \frac{|x_0^*(x)|}{\|x_0^*\|} = \frac{\|x\|}{1} = \|x\|$$

(6) X normed space

$Y \subseteq X$ subspace, dense $\Leftrightarrow$ "$\forall x^* \in X^*, x^*(y) = 0\ \forall y \in Y \Rightarrow x^* = 0$"

Pf: "$\Rightarrow$" $\forall x \in X$, $\exists\, y_n \in Y \ni y_n \to x$

$$\Rightarrow \underset{\underset{0}{\parallel}}{x^*(y_n)} \to x^*(x)$$

$$\Rightarrow x^*(x) = 0\ \forall x \in X$$

"$\Leftarrow$" Assume $\overline{Y} \neq X$

Let $x_0 \in X \setminus \overline{Y}$. Then $\inf\limits_{y \in \overline{Y}} \|y - x_0\| > 0$.

Then $(1) \Rightarrow \exists\, x^* \in X^* \ni x^*(x_0) = 1,\ x^*(Y) = 0$

$$\Downarrow$$

$$x^* = 0 \ \rightarrow\leftarrow.$$

X normed space

$x^* \in X^*$, $x^* \neq 0$

$\because \ker x^* = \left\{ x \in X : x^*(x) = 0 \right\} \neq X$

Let $x_0 \in X$, but $x_0 \notin \ker x^*$

Then $\forall x \in X$, $x = z + \lambda x_0$, where

$$z \in \ker x^* \ \&\ \lambda \in F \ \&\ \text{uniquely, i.e., } \dim(X / \ker x^*) = 1$$

Pf: Let $\lambda = x^*(x) / x^*(x_0)$

Then $z = x - \lambda x_0 \in \ker x^*$

(Reason: $x^*(z) = x^*(x) - \dfrac{x^*(x)}{x^*(x_0)} \cdot x^*(x_0) = 0$)

Say, $z + \lambda x_0 = 0 \Rightarrow \underset{\underset{\ker x^*}{\cap}}{z} = -\lambda x_0 \Rightarrow \lambda = 0 \Rightarrow z = 0$

Geometrical interpretation:

Explanation:

For $x^* \neq 0$, $\ker x^*$ hyperplane

X normed space

$x^* \in X^*, x^* \neq 0, c \in \mathrm{R}$

$X = \mathrm{R}^n$

$\left(\begin{array}{l} \text{Idea: Consider } x^* = [a_1, \ldots, a_n] \\ \qquad \therefore x^*(x) = c \leftrightarrow a_1 x_1 + \ldots + a_n x_n = c \end{array} \right)$

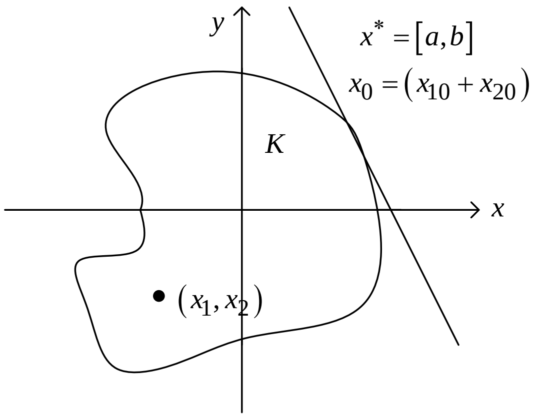
Def. $\left\{ x \in X : \operatorname{Re} x^*(x) = c \right\}$ (hyperplane determined by $\underset{\underset{\text{slope}}{\uparrow}}{x^*}$ & $\underset{\underset{\text{location}}{\uparrow}}{c}$)

Def. $K \subseteq X$, $x_0 \in K$, $x^* \neq 0 \in X^*$

$\left\{ x \in X : \operatorname{Re} x^*(x) = \operatorname{Re} x^*(x_0) \right\}$ tangent hyperplane to K at x_0

(determined by x^* & $\operatorname{Re} x^*(x_0)$)

if $\operatorname{Re} x^*(x) \leq \operatorname{Re} x^*(x_0)$ $\forall x \in K$



Ex. $X = \mathrm{R}^2$

$ax_1 + bx_2 = c$

$ax_1 + bx_2 \le ax_{10} + bx_{20} \ \ \forall (x_1, x_2) \in K$

$[a_1, a_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c$

$\Updownarrow$

$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = c,$ where $f \in \mathrm{R}^{2*}$

(7) X normed space

$K = \{x \in X : \|x\| \le 1\}, \ \|x_0\| = 1$

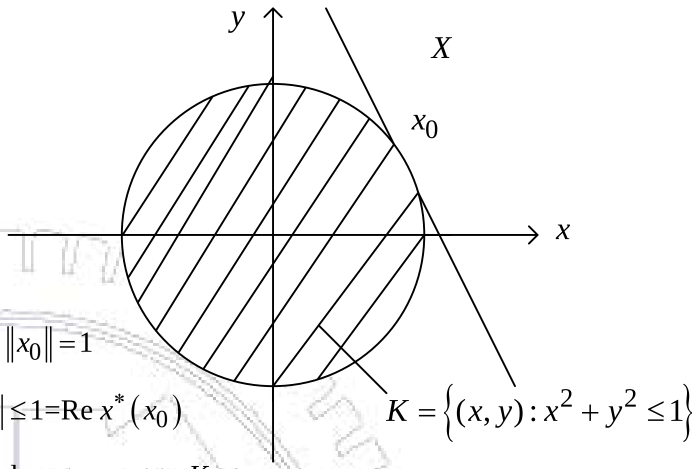
Then $\exists$ tangent hyperplane to K at x_0

Pf. $\because x_0 \neq 0$

(2) $\Rightarrow \exists\, x^* \in X^* \ni \ \|x^*\| = 1$ & $x^*(x_0) = \|x_0\| = 1$

$\therefore \forall\, x \in K, \operatorname{Re} x^*(x) \le |x^*(x)| \le \|x^*\| \cdot \|x\| \le 1 = \operatorname{Re} x^*(x_0)$

$\therefore x^*$ & $\operatorname{Re} x^*(x_0)$ determine the hyperplane tangent to K at x_0



Homework:

Sec.4.8: Ex.4.8.4, Ex.4.8.7 (need: Ex.4.8.5 & 4.8.6)