Class 53

X normed space over $F=$ R or C

Def. $\langle\cdot,\cdot\rangle : X\times X^* \to F$ (outer product)

$\ni \left\langle x,x^*\right\rangle = x^*\left(x\right)$

Then (1) $\left\langle ax+by,x^*\right\rangle = a\left\langle x,x^*\right\rangle + b\left\langle y,x^*\right\rangle$.

(2) $\left\langle x,ax^*+by^*\right\rangle = a\left\langle x,x^*\right\rangle + b\left\langle x,y^*\right\rangle$ (bilinear)

(3) $\left|\left\langle x,x^*\right\rangle\right| \le \|x\|\cdot\left\|x^*\right\|$ (Schwarz $\le$)

(4) $x=0 \Leftrightarrow \left\langle x,x^*\right\rangle = 0\ \forall x^*\in X^*$

Pf.: "$\Leftarrow$" by (2) on p.19

(5) $x^*=0 \Leftrightarrow \left\langle x,x^*\right\rangle = 0\ \forall x\in X$

Note: Difference with inner product in Hilbert space:

(2) $\leftrightarrow \langle z,ax+by\rangle = \bar{a}\langle z,x\rangle + \bar{b}\langle z,y\rangle$ (sesquilinear)

$1\frac{1}{2}$

$S\subseteq X$, subset

Def. $S^{\perp} = \left\{x^*\in X^* : x^*\left(x\right)=0\ \forall x\in S\right\}$ (ortho. complement of S)

$S^*\subseteq X^*$

Def. $S^{*\perp} = \left\{x\in X : x^*\left(x\right)=0\ \forall x^*\in S^*\right\}$ (ortho. complement of S^*)

Properties:

(1) $\forall\ S\subseteq X$, $S^{\perp}$ closed subspace of X^*

(2) $\forall\ S^*\subseteq X^*$, $S^{*\perp}$ closed subspace of X.

(3) $S\subseteq T\subseteq X \Rightarrow T^{\perp}\subseteq S^{\perp}$

(4) $S^*\subseteq T^*\subseteq X^* \Rightarrow T^{*\perp}\subseteq S^{*\perp}$

(5) $\forall S\subseteq X, \left(S^{\perp}\right)^{\perp} =$ closed linear span of S

(Let $T=$ losed linear span of S

$\because\ S\subseteq T\Rightarrow S^{\perp}\supseteq T^{\perp}\Rightarrow\left(S^{\perp}\right)^{\perp}\subseteq\left(T^{\perp}\right)^{\perp}=T$, if T is a closed subspace)

(6) $\forall S^*\subseteq X^*, \left(S^{*\perp}\right)^{\perp} =$ closed linear span of S^*

(7) $X^{\perp}=\{0\}\subseteq X^*$ (by (5) above)

(8) $X^{*\perp}=\{0\}\subseteq X$ (by (4) above)

$$\left.\begin{aligned}&(9)\ \{0\}^{\perp}=X\\ &(10)\ \{0\}^{\perp}=X^*\end{aligned}\right\}\text{(by (3), (4), (5), (6))}$$

Sec.4.9

Dirichlet problem:

Find u on $\Omega \subseteq \mathrm{R}^n \ni$

$$\begin{cases} \mathrm{V}u = \sum_{j=1}^{n} \dfrac{\partial^2 u}{\partial xj^2} = 0 \text{ on } \Omega \text{ (Laplace equation)} \\ u = f \text{ on } \partial\Omega \end{cases}$$

Thm. When $n = 2$ & under certain conditions on Ω, Green's func. exists.

Pf.: Use Hahn-Banach Thm

(Ex.4.9.2) any solu. u to Dirichlet problem can be expressed as an integral of f & Green's function

Sec.4.10. Reflexive spaces

Spaces: duality theory (Sec.4.8 ~ 4.14): Hahn-Banach Thm
Operators: spectral theory (Chap.5, 6): compact,normal.

Thm. X normed space

$$X^* \text{ separable} \underset{\not\Leftarrow}{\Rightarrow} X \text{ separable}$$

Note: 1. "$\not\Leftarrow$": l^1 separable, but $l^{1*} \cong l^\infty$ nonsep.

2. This says "X^* larger than X"

l^1 sep. (Ex.3.1.6):

Reason: $\{(x_1,\ldots,x_n,0,\ldots) : n \ge 1, x_j\text{'s rational}\}$ dense in l^1

$$\because \aleph_0 + \aleph_0 \cdot \aleph_0 + \ldots = \aleph_0 + \aleph_0 + \ldots = \aleph_0$$

l^∞ nonsep. (Ex.3.1.7):

Reason: $\{(x_1, x_2, \ldots) : x_j\text{'s rational}\}$ dense in l^∞

$$\aleph_0 \cdot \aleph_0 \ldots = \aleph_0^{\aleph_0} \ge 2^{\aleph_0} = \aleph_1 > \aleph_0$$

Pf.: Let $\{x_n^*\}$ dense in X^*

$$\because \|x_n^*\| = \sup_{\|x\|=1} |x_n^*(x)| > \frac{1}{2}\|x_n^*\| \text{ if } x_n^* \ne 0$$

$$\Rightarrow \exists\, x_n \in X \ni \|x_n\| = 1 \;\&\; |x_n^*(x_n)| \ge \frac{1}{2}\|x_n^*\|$$

(In Hilbert space, this means x_n^* & x_n close to each other)

Let $A = \{$finite linear combinations of x_n with rational coeffi.$\}$

Then A countable

Check: $\overline{A} = X$.

Assume $\overline{A} \neq X$

Cor.4.8.7 $\Rightarrow \exists\, x^* \neq 0 \ \ni\ x^*(y) = 0 \ \forall y \in \overline{A}$

$\because \left\{x_n^*\right\}$ dense in X^*

$\Rightarrow \exists \left\{x_{n_k}^*\right\} \ \ni\ x_{n_k}^* \to x^*$ in norm

$$\therefore \frac{1}{2}\left\|x_{n_k}^*\right\| \le \left|x_{n_k}^*\left(x_{n_k}\right)\right| = \left|x_{n_k}^*\left(x_{n_k}\right) - x^*\left(x_{n_k}\right)\right| \le \left\|x_{n_k}^* - x^*\right\| \cdot \underset{\overset{\parallel}{1}}{\left\|x_{n_k}\right\|} \to 0$$

$\Rightarrow x_{n_k}^* \to 0$ in norm

$\Rightarrow x^* = 0 \ \to\leftarrow$

Reflexivity:

$X \to X^* \to X^{**}$

$k : x \mapsto \hat{x}$

$$\hat{x}\left(x^*\right) = x^*(x) \ \ \forall x^* \in X^* \ \left(\begin{array}{l} \text{i.e., } \left\langle x^*, \hat{x}\right\rangle = \left\langle x, x^*\right\rangle \ \ \forall x^* \in X^* \\ \quad \text{similar to inner product} \end{array}\right)$$

Note: X^* always Banach space (by Thm 4.4.4)

Thm. X normed space.

Then $k : X \to X^{**}$ isometric isom. from X into X^{**} & if X Banach space, then $k(X)$ is closed in X^{**}

Pf: Check: (1) $\hat{x}$ bdd linear functional on X^* ($\because \left|\hat{x}\left(x^*\right)\right| \le \left\|x^*\right\| \cdot \|x\| \Rightarrow \|\hat{x}\| \le \|x\|$).

(2) k linear

(3) $\|\hat{x}\| = \|x\|$:

$$\because \|x\| = \sup_{\left\|x^*\right\|=1} \left|x^*(x)\right| = \sup_{\left\|x^*\right\|=1} \left|\hat{x}\left(x^*\right)\right| \le \|\hat{x}\| \cdot \underset{\overset{\parallel}{1}}{\left\|x^*\right\|}$$

↑ (p.153, Cor.4.8.6)

(4) kX closed in X^{**} if X Banach space

Pf: Assume $\hat{x}_n \to y$ in X^{**}

$\because \left\|x_n - x_m\right\| = \left\|\hat{x}_n - \hat{x}_m\right\| < \varepsilon$ if n, m large

$\therefore x_n \to x$ in X

$\Rightarrow \hat{x}_n \to \hat{x}$ in X^{**}

But $\hat{x}_n \to y$

$\Rightarrow y = \hat{x} \in kX$

Another proof: $kX \cong X$ Banach space $\Rightarrow kX$ closed in X^{**}

Application:

Thm. X normed space

$\{x_\alpha\} \subseteq X$

Then $\left\{\left|x^*(x_\alpha)\right|\right\}$ bdd $\forall x^* \in X^* \Rightarrow \{\|x_\alpha\|\}$ bdd

Note: "$\Leftarrow$" trivial

Pf.: Apply uniform bddness principle to $\left\{\hat{x}_\alpha : X^* \to F\right\}$ ($\because X^*$ Banach space)

$$\text{Then } \Big\{\underset{\underset{|x^*(x_\alpha)|}{\parallel}}{\left|\hat{x}_\alpha\left(x^*\right)\right|}\Big\} \text{ bdd } \forall x^* \in X^* \Rightarrow \Big\{\underset{\underset{\|x\|}{\parallel}}{\|\hat{x}_\alpha\|}\Big\} \text{ bdd}$$

Def. X $\underline{\text{normed space}}$ is reflexive if $k(X) = X^{**}$

(i.e., $X \cong X^{**}$ under the natural embedding k)

isometric isom.

Note 1. X reflexive $\Rightarrow X$ Banach space

Note 2. X reflexive $\underset{\not\Leftarrow}{\Rightarrow} X \cong X^{**}$

Ex. X finite-dim normed space ($\therefore \dim X^{**} = \dim X^* \underset{\downarrow}{=} \dim X \Rightarrow k : X \to X^{**}$ 1-1$\Rightarrow$ must be onto)

Let $e_1, \ldots, e_n$ basis of X. Let $x_i^*\left(e_j\right) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$

Then $x_1^*, \ldots, x_n^*$ basis of X^*

l^p reflexive iff $1 < p < \infty$

c_0, l^1, l^∞, $C[a,b]$ not reflexive

Hilbert spaces are reflexive

Prop 1. X reflexive

Then X separable iff X^* separable.

Pf.: "$\Rightarrow$" $\because X \cong X^{**}$ separable

$\Rightarrow X^*$ separable

"$\Leftarrow$" proved before.

Prop 2. X, Y normed spaces

Then $X \cong Y$

(isometric isom.)

Then X reflexive $\Leftrightarrow Y$ reflexive

Pf. Omitted