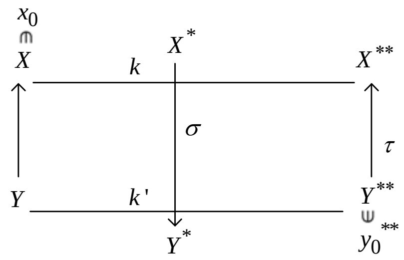
Class 54

Prop. 3. X reflexive Banach space

$Y \subseteq X$ closed subspace

$\Rightarrow Y$ reflexive

Pf.:



Check: k onto $\Rightarrow k'$ onto.

(1) Let $\sigma: X^* \to Y^*$

$\sigma\left(x^*\right) = x^*|Y \;\; \forall x^* \in X^*$

Then $\sigma\left(x^*\right) \in Y^*$ $(\because \|\sigma\| \le 1)$

(2) Let $\tau: Y^{**} \to X^{**}$

$\left(\tau\, y^{**}\right)\left(x^*\right) = y^{**}\left(\sigma x^*\right) \;\; \forall x^* \in X^*$

Then $\tau\, y^{**} \in X^{**}$

$$\left(\begin{array}{l} \because \left|y^{**}\left(\sigma x^*\right)\right| \le \left\|y^{**}\right\| \cdot \left\|\sigma\left(x^*\right)\right\| \\ \quad \le \left\|y^{**}\right\| \cdot \left\|x^*\right\| \\ \quad \Rightarrow \left\|\tau\, y^{**}\right\| \le \left\|y^{**}\right\| \\ \quad \Rightarrow \|\tau\| \le 1 \end{array}\right)$$

Let $y_0^{**} \in Y^{**}$

Let $x_0 = k^{-1}\left(\tau\, y_0^{**}\right) \in X$

Check: (3) $x_0 \in Y$ & (4) $k'\left(x_0\right) = y_0^{**}$

(3) Assume $x_0 \notin Y$

Hahn-Banach Thm $\Rightarrow \exists\, x^* \in X^* \ni x^*(x_0) \neq 0$ & $x^*(Y) = 0$

$$\Downarrow (1)$$

$$\sigma(x^*) = 0$$

$$\Downarrow (2)$$

$$\tau(y^{**})(x^*) = 0 \quad \forall y^{**} \in Y^{**}$$

$$\Downarrow \text{ in parti.}$$

$$\tau(y_0^{**})(x^*) = 0$$

$$\Downarrow \text{ by def. of } x_0$$

$$k(x_0)(x^*) = 0$$

$$\Downarrow \text{ by def. of } k$$

$$x^*(x_0) = 0$$

$$\rightarrow\leftarrow$$

Hence $x_0 \in Y$

(4) Check: $(k' x_0)(y^*) = y_0^{**}(y^*)\ \forall y^* \in Y^*$

$\because k'(x_0)(y^*) = y^*(x_0)$

Hahn-Banach $\Rightarrow$ Extend y^* to x^* on X preserving norm

$$y^*(x_0) = x^*(x_0)$$

$$\| \text{ def. of } k$$

$$k(x_0)(x^*)$$

$$\| \text{ def. of } x_0$$

$$\tau(y_0^{**})(x^*)$$

$$\| \ (2)$$

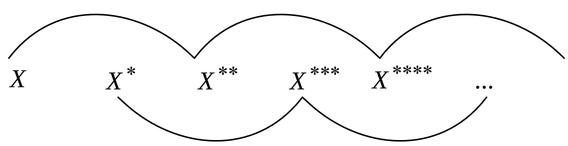
$$y_0^{**}(\sigma x^*)$$

$$\| \ (1)$$

$$y_0^{**}(y^*)$$

Prop. 4 X Banach space

Then X reflexive iff X^* reflexive



Note. Prop. 4 $\Rightarrow$ any one reflexive then all reflexive

Pf.: "$\Rightarrow$":

Check: $X^* \overset{k}{\to} X^{***}$ onto

$\forall x^{***}$, $X^{**} \overset{x^{***}}{\to} F \Rightarrow x^{***}k \in X^*$

$\uparrow k$

X

Check: $x^{***} = \overbrace{x^{***}k}$

$\Rightarrow X^*$ reflexive

"$\Leftarrow$":

X^* reflexive

$\Rightarrow X^{**}$ reflexive (by above "$\Rightarrow$")

$\because k(X)$ closed subspace of X^{**} ($\because X$ Banach space)

$\Rightarrow k(X)$ reflexive

$\cong$

$\Rightarrow X$ reflexive

Def. $x_n, x \in X$

$x_n \to x$ weakly if $x^*(x_n) \to x^*(x) \ \forall x^* \in X^*$

$x_n \to x$ strongly or in norm if $\|x_n - x\| \to 0$

Note.1. $x_n \to x, x_n \to y$ weakly $\Rightarrow x = y$ (Ex.4.10.1)

Pf: $\forall x^* \in X^*$, $x^*(x_n) \to x^*(x)$

$\to x^*(y)$

$\Rightarrow x^*(x) = x^*(y) \ \forall x^* \in X^*$

$\Rightarrow x = y$ (Hahn-Banach Thm)

2. $x_n \to x$ strongly $\underset{\not\Leftarrow}{\Rightarrow} x_n \to x$ weakly

Pf: $|x^*(x_n) - x^*(x)| \le \|x^*\| \cdot \|x_n - x\| \to 0 \ \forall x^* \in X^*$

Ex. Let $x_n = (0...0,1,0,...)$ in l^2

$$\uparrow \text{nth}$$

$x = (0,0,...)$

Then $\forall\, y \in l^2, \langle x_n, y\rangle = y_n \to 0 = \langle x, y\rangle$

$\quad \| $

$(y_1, y_2, ...)$

i.e., $x_n \to x$ weakly

But $\|x_n\| = 1 \not\to 0 \quad \forall n$

Riesz: $f \in l^{2*} \Leftrightarrow \exists y \in l^2 \ni f(x) = \langle x, y\rangle \quad \forall x^* \in l^2$

Note: $\{x \in l^2 : \|x\| = 1\}$ is not weakly closed.

$\{x \in l^2 : \|x\| = 1\}$ is norm closed.

3. X finite-dim normed space

Then $x_n \to x$ strongly $\Leftrightarrow x_n \to x$ weakly (Ex.4.10.3)

4. Weak convergence topology $\left(\begin{array}{l} \because X \cong kX \subseteq X^{**} \\ \therefore \text{induced from weak top}^* \text{ on } X^{**} \end{array}\right)$

In general, not metrizable

$\left\{\begin{array}{l} \text{weakly sequentially compact} \leftrightarrow \text{weakly compact} \\ \text{weakly closed} \\ \text{weak Cauchy} \\ \text{weakly complete.} \end{array}\right.$

All defined in terms of sequences (instead of nets)

5. Two topologies are different:

(I) (Ex.4.10.4) $S \subseteq X$ weakly closed $\Rightarrow S$ closed ($\not\Leftarrow$)

Pf: $x_n \to x$ strongly & $x_n \in S$

$$\Downarrow$$

$x_n \to x$ weakly

$\Rightarrow x \in S$

Ex. $S = \{x \in l^2 : \|x\| = 1\}$ closed, but not weakly closed

(II) $S \subseteq X$ sequentially compact $\Rightarrow$ weakly sequentially compact ($\not\Leftarrow$) (Ex.4.10.5)

(III) X weakly complete $\Rightarrow$ complete ($\not\Leftarrow$) (Ex.4.10.7)

Thm. X normed space

$x_n \to x$ weakly

Then (1) $\{x_n\}$ bdd;

(2) $x \in$ closed linear span of $\{x_n\}$;

(3) $\|x\| \le \underline{\lim}\|x_n\|$

$$\text{Note: } \begin{cases} \text{(1) true if } x_n \to x \text{ strongly;} \\ \text{(2) If } x_n \to x \text{ strongly, then } x \in \overline{\{x_n\}}. \\ \text{(3) says: } x \mapsto \|x\| \text{ is lower semi-conti. w.r.t. weak top.} \end{cases}$$

Ex. $x_n = \Big(0,\ldots,0,\underset{nth}{1},0,\ldots\Big) \to 0$ weakly in l^2

$\Rightarrow \underset{\substack{\Downarrow \\ 0}}{\|0\|} < \underline{\lim}\|x_n\| = 1.$