Class 57

Pf. (1) $\because \left\{x^*(x_n)\right\}$ conv. $\Rightarrow$ bdd $\forall x^* \in X^*$

$\Rightarrow \{\|x_n\|\}$ bdd ($\because$ unif. bddness principle $\because X^*$ is Banach space)

(2) Let $Y =$ closed linear span of $\{x_n\}$

Assume $x \notin Y$

Then $\exists\, x^* \in X^* \to x^*(x) \neq 0$ & $x^*(x_n) = 0\ \forall n \geq 1$ (Hahn-Banach Thm)

$$\downarrow$$

$$x^*(x) = 0 \ \rightarrow\leftarrow$$

(3) $\because \|x\| \overset{\text{Hahn-Banach}}{=} \sup_{\|x^*\|=1} \left|x^*(x)\right| \leq \underline{\lim}\|x_n\|$

$$\uparrow$$

$$\left|x^*(x_n)\right| \leq \underbrace{\|x^*\|}_{=1} \cdot \|x_n\| \Rightarrow \underbrace{\lim_n \left|x^*(x_n)\right|}_{=\left|x^*(x)\right|} \leq \underline{\lim}_n \|x_n\|$$

Thm

X reflexive Banach space
$K \subseteq X$ weakly sequentially compact $\Leftrightarrow$ K bdd & weakly closed

Note 1. Generalize: Bolzano-Weierstrass Thm in dim $X < \infty$:

Note 2. X normed space "K bdd, closed $\Leftrightarrow$ K compact" $\Leftrightarrow$ dim $X < \infty$ (cf. p.133)

Note 3. X Banach space, $K \subseteq X$

Then K weakly compact $\Leftrightarrow$ K weakly sequentially compact.

(Eberlein-Smulian Thm)

Ref. J.B.Conway, A course in functional analysis, 2nd.,p.163. Thm.V.13.1.

Note 4. X Banach space, Then X reflexive

$\Leftrightarrow \{x \in X : \|x\| \leq 1\}$ weakly compact

Ref. J.B.Conway, p.132, Thm.V.4.2.

Pf.: "$\Rightarrow$" (valid $\forall$ normed space X)

Let K be weakly sequentially compact.

(1) K bdd:

Assume otherwise.

Then $\forall n$, $\exists\, x_n \in K \ni \|x_n\| \geq n$.

K weakly sequentially compact $\Rightarrow \exists\, x_{n_k} \ni x_{n_k} \to x \in K$ weakly

From preceding thm (1), $\left\{x_{n_k}\right\}$ bdd

But $\left\|x_{n_k}\right\| \geq n_k \ \rightarrow\leftarrow$

(2) K weakly closed:

Let $x_n \in K \ni x_n \to x$ weakly.

Then $\exists\, x_{n_k} \to y \in K$ weakly

But $x_{n_k} \to x$

$\Rightarrow x = y \in K$

$\therefore K$ closed

" $\Leftarrow$ "

Let K be bdd & weakly closed.

Let $\{x_n\} \subseteq K$

Check $\exists \{x_{n_j}\}$ converges weakly

Let $Y = \overline{\vee \{x_n\}}$

Then Y separable closed subspace of reflexive X

$\Rightarrow Y$ reflexive (Thm 4.10.5)

$\because Y^{**} \cong Y$ separable

$\Rightarrow Y^*$ separable (Thm 4.10.1) Let $Y^* = \overline{\{x_n\}}$

Bolzano-Weierstrass Thm in $F \Rightarrow$

diagonalization
$$\begin{cases} \because \left|x_1^*(x_n)\right| \le \left\|x_1^*\right\| \cdot \left\|x_n\right\| \text{ bdd in } F \\ \Rightarrow \exists\, x_1^*(x_{n,1}) \text{ convergent} \\ \because \left|x_2^*(x_{n,1})\right| \le \left\|x_2^*\right\| \cdot \left\|x_{n,1}\right\| \text{ bdd in } F \\ \Rightarrow \exists\, x_2^*(x_{n,2}) \text{ convergent} \\ \qquad \vdots \\ \Rightarrow \exists\, x_k^*(x_{n,k}) \text{ convergent} \\ \qquad \vdots \end{cases}$$

Let $y_k = x_{k,k}$ subseq. of x_n

Check: y_k weakly convergent, i.e., $x^*(y_k)$ converges $\forall x^* \in X^*$.

(i) Check: $z^*(y_k)$ converges $\forall z^* \in Y^*$ (Ex.4.10.2)

$\because x_n^*(y_k)$ converges $\forall n$

Let $z^* \in Y^*$

$\because \{x_n^*\}$ dense in Y^*

$\therefore \exists\, x_n^* \ni \left\|z^* - x_n^*\right\| < \varepsilon$

$$\therefore \left|z^*(y_k) - {}^*(y_j)\right| \le \left|z^*(y_k) - x_n^*(y_k)\right| + \left|x_n^*(y_k) - x_n^*(y_j)\right| + \left|x_n^*(y_j) - z^*(y_j)\right|$$

$$\le \left\|z^* - x_n^*\right\| \cdot \|y_k\| + \left|x_n^*(y_k) - x_n^*(y_j)\right| + \left\|x_n^* - z^*\right\| \cdot \|y_j\|$$

$$< \varepsilon \cdot M + \left|x_n^*(y_k) - x_n^*(y_j)\right| + \varepsilon \cdot M$$

$\therefore$ small for large k, j

$\therefore$ Cauchy

$\Rightarrow$ $z^*(y_k)$ converges

(ii) Check: $\exists\, y \in z \ni z^*(y_k) \to z^*(y) \ \forall z^* \in Y^*$

$\because \hat{y}_k(z^*)$ converges $\forall z^* \in Y^*$

Thm.4.5.2 $\Rightarrow \exists\, y^{**} \in Y^{**} \ni \hat{y}_k(z^*) \to y^{**}(z^*) \ \forall z^* \in Y^*$

(i.e., unif. bddness principle) $\quad \hat{y}_k(z^*) = z^*(y_k), \quad y^{**}(z^*) = z^*(y)$ ($\because Y$ reflexive $\Rightarrow y^{**} = \hat{y}$ for some $y \in Y$)

(iii) Check: $x^*(y_k) \to x^*(y) \ \forall x^* \in X^*$

Let $z^* = x^*|Y$

$\therefore z^*(y_k) \to z^*(y)$,i.e., $y_k \to y$ weakly $\therefore K$ weakly closed $\Rightarrow y \in K$

$z^*(y_k) = x^*(y_k), \quad z^*(y) = x^*(y)$