Class 58

Sec.4.11. Topology

X_α top. spaces

Let $X = \prod\limits_{\alpha} X_\alpha$

Then $x_n = \{x_n(\alpha)\} \to x = \{x(\alpha)\} \Leftrightarrow x_n(\alpha) \to x(\alpha)\ \forall \alpha$

Tychonoff Thm: X compact $\Leftrightarrow$ X_α compact $\forall \alpha$

Pf.: "$\Rightarrow$" easy (Ex.4.11.4)

"$\Leftarrow$" Use Zor'ns Lemma

Note: Not counterintuitive because product top. definition

$\because$ open set few

$\Rightarrow$ open covering has finite covering

Sec.4.12 Weak Topology in X^*($\Rightarrow$ Top. vector space, not normed space)

(weak-$*$ top)

X: norm (strong) top., weak top.

X^*: norm top., weak-$*$ top.

Def. $x_n^* \to x^*$ in norm if $\left\| x_n^* - x^* \right\| \to 0$ (unif. conv.)

Def. $x_n^* \to x^*$ weak-$*$ if $\forall x \in X, \left| x_n^*(x) - x^*(x) \right| \to 0$ (pointwise conv.)

X normed space

$$\forall x^* \in X^*,\ \text{let } W(x^*) = \left\{ \overbrace{\left\{ y^* : \left| y^*(x_i) - x^*(x_i) \right| < \varepsilon\ \forall_{1 \le i \le n}, x_n \in X \right\}}^{N(x^*; x_1, \ldots, x_n; \varepsilon)} \right\}$$

$\Updownarrow$

$\forall x \in X$, associate one F

Then $\prod\limits_{x \in X} F$ product top. $\Leftrightarrow$ weak top. on X^*

V/

$Y \cong X^*$ $\to$ Y linear in indices

Note 1. $x_n^* \to x^*$ weakly in X^*

$\Leftrightarrow \forall$ nbd of x^*, $W\left(x^*\right)$, $\exists N \ni n \geq N \Rightarrow x_n^* \in W\left(x^*\right)$

$\Updownarrow$

$\Leftrightarrow \forall \varepsilon > 0, \forall x_1,\ldots,x_m \in X, \exists N \ni n \geq N \Rightarrow \left|x_n^*(x_i) - x^*(x_i)\right| < \varepsilon$ for $1 \leq i \leq m$

$\Leftrightarrow x_n^*(x) \to x^*(x) \ \forall x \in X$

Note 2. $x_n^* \to x^*$ in norm $\underset{\not\Leftarrow}{\Rightarrow}$ $x_n^* \to x^*$ weakly

Pf.: $\left|x_n^*(x) - x^*(x)\right| \leq \left\|x_n^* - x^*\right\| \cdot \|x\| \to 0 \ \forall x \in X$

Ex. $X = l^2$

Let $x_n^*(x_1,\ldots,x_n,\ldots) = x_n,\ n \geq 1$

Then $x_n^* \to 0$

(Reason: $\forall x \in l^2$, $x_n^*(x) = x_n \to 0$)

But $\left\|x_n^*\right\| = 1 \not\to 0$

i.e., $x_n^* \not\to 0$ in norm

Note 3. "$x_n^* \to x^*$ weak-$*$" means "pointwise conv."

"$x_n^* \to x^*$ in norm" means "unif. conv."

Note 4. $\dim X < \infty$, $x_n^*,\ x^* \in X^*$

Then $x_n^* \to x^*$ in norm $\Leftrightarrow x_n^* \to x^*$ weak-$*$ (Ex.)

Alaoglu Thm: ($\leftrightarrow$ Thm.4.10.8)

X normed space, over F

Then $B \equiv \left\{ f^* \in X^* : \left\|f^*\right\| \leq 1 \right\}$ is weak-$*$ compact

Note: Generalizes Bolzano-Weierstrass Thm in $\dim X < \infty$; not true for norm top.

Pf.: $\forall x \in X$, let $I_x = \begin{cases} \left[-\|x\|, \|x\|\right] & \text{if } F = \mathbb{R} \\ \left\{z \in \mathbb{C} : |z| \leq \|x\|\right\} & \text{if } F = \mathbb{C} \end{cases}$

Let $I = \prod_{x \in X} I_x$ compact

Define $\underset{\substack{\cap \\ B \\ \text{weak-}*\text{ top.}}}{f^*} \mapsto \underset{\substack{\cap \\ I \\ \text{product top.}}}{\left\{ f^*(x) \right\}} \equiv \hat{f}^*$

$\left(\because \left| f^*(x) \right| \le \left\| f^* \right\| \cdot \|x\| \le \|x\| \right)$

Then $\underset{\uparrow}{\text{1-1}}$, & both $\underset{\uparrow}{\text{conti.}}$

$\because \hat{f}_1^* = \hat{f}_2^* \Leftrightarrow f_1^*(x) = f_2^*(x) \ \forall x$
$\Updownarrow$
$f_1^* = f_2^*$

$\because f_n^* \to f^*$ weak-* $\Leftrightarrow f_n^*(x) \to f^*(x) \ \forall x \in X$
$\Updownarrow$
$\left\{ f_n^*(x) \right\} \to \left\{ f^*(x) \right\}$
$\Downarrow \qquad \Downarrow$
$\hat{f}_n^* \qquad \hat{f}^*$