Class 59

Let $I' = \left\{ \hat{f}^* : f^* \in B \right\} \subseteq I$

Check: I' compact ($\Rightarrow B \cong I'$)

Check: I' closed ($\because$ closed subset of compact I is compact)

Let $\eta \in \overline{I'}$

Check: $\exists f_0^* \in B \ni \eta = \hat{f}_0^*$

Let $f_0^* : X \to F$ be $\ni f_0^*(x) = \eta_x$

$\because \exists \hat{f}_n^* \to \eta$ in product top.

$\Rightarrow f_n^*(x) \to \eta_x \ \forall x \in X$

(i) Check: f_0^* linear

$$\left.\begin{aligned} \therefore \lambda_1 x_n^*(x_1) &\to \lambda_1 \eta_{x_1} \\ \lambda_2 x_n^*(x_2) &\to \lambda_2 \eta_{x2} \end{aligned}\right\} \to \lambda_1 x_n^*(x_1) + \lambda_2 x_n^*(x_2) \to \lambda_1 \eta_{x_1} + \lambda_2 \eta_{x_2} = \lambda_1 f_0^*(x_1) + \lambda_2 f_0^*(x_2)$$

$$x_n^*(\lambda_1 x_1 + \lambda_2 x_2) \to \eta_{\lambda_1 x_1 + \lambda_2 x_2} = f_0^*(\lambda_1 x_1 + \lambda_2 x_2)$$

(ii) Check: $\left\| f_0^* \right\| \le 1 : \left| f_0^*(x) \right| = |\eta_x| \le \|x\|$ ($\because \eta \in \overline{I'} \subseteq I$) $\Rightarrow f_0^* \in B$

(iii) Check: $\hat{f}_n^* = \eta$

$\because f_0^* = \left\{ f_0^*(x) \right\} = \{\eta_x\} = \eta$

$\Rightarrow \eta \in I'$

$\therefore \overline{I'} = I'$ is closed

Thm. 4.12.3. Note: This is for general K insteady of ball

X separable normed space Then the following are equiv.: (1) $K \subseteq X^*$ weakly sequen. compact; (2) K bdd & weakly closed; (3) K weakly compact.

Note: Generalize Bolzano-Weierstiass thm for dim $X < \infty$.

Pf.: (1) $\Rightarrow$ (2) as before for any normed space

(2) $\Rightarrow$ (3) by Alaoglu thm & for any normed space

(3) $\Rightarrow$ (1): Let $\{y_n\}$ dense in X

Consider $\left\{K \cap N\left(z^*; y_{k_1}, \ldots, y_{k_n}, \frac{1}{m}\right)\right\}$ nbd base at $z^* \in K$

$\because$ countable

$\therefore$ K first countable (i.e. countable nbd base at each point of K)

$\therefore$ K weakly compact $\Rightarrow$ weakly sequentially compact

Homework:

Sec.4.12, Ex.2,5

(Thm. 4.11.5)

Note: More generally, X first countable top. space, $K \subseteq X$

Then K compact $\Rightarrow$ K sequentially compact