Class6

Section 1.7. Metric space

X set

Def. $\rho: X \times X \to \mathbb{R}$ is a metric if Then (X,ρ) metric space.

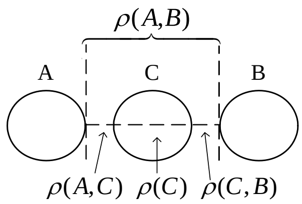
(1) $\rho(x,y) \geq 0$

(2) $\rho(x,y) = 0 \iff x = y$

(3) $\rho(x,y) = \rho(y,x)$ (symmetry)

(4) $\rho(x,y) \leq \rho(x,y) + \rho(y,z)$ $(\Delta \leq)$.

Ex.



Def.1. $A, B \subseteq X$

$\rho(A,B) = \inf_{x\in A,\, y\in B} \rho(x,y)$. (distance between A&B).

Def.2. $A \subseteq X$, $x \in X$

$\rho(x,A) = \inf_{y\in A} \rho(x,y)$

Def.3. $A \subseteq X$

$d(A) = \sup_{x,y\in A} \rho(x,y)$ (diameter of A)

Def. $A \subseteq X$ is bdd if $d(A) < \infty$.

Note.1. For $A, B, C \subseteq X$, $\rho(A,B) \leq \rho(A,C) + \rho(C,B) + d(C)$

(1)&(3) hold, but not (2)

Note.2. $\rho(A,B) \leq \rho(A,x) + \rho(x,B)$

Note.3. $\rho(x,y) \leq \rho(x,A) + \rho(A,y)$

Topological concepts:

open set, closed set,

interior, accumulation pt., closure (for an arbitrary set)

convergent sequence, limit, Cauchy seq.

Def. Metric space (X,ρ) is complete if every Cauchy seq. converges.

Def. $B = \sigma$-algebra generated by open sets of X

$= \sigma$-ring generated by open sets of X

$= \sigma$-ring generated by closed sets of X

Borel sets $\to$ contains open sets, closed sets etc.

Homework: 1.7.8

Relating measure to metric:

Sec.1.8. (X,ρ) metric space

u^* outer measure on X

Def. u^* metric outer measure if $A, B \subseteq X \ni \rho(A,B) > 0$

$\Rightarrow u^*(A \cup B) = u^*(A) + u^*(B)$.

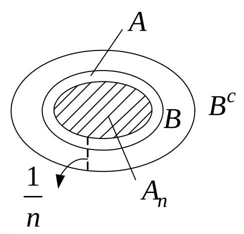
Note: $\rho(A,B) > 0 \overset{\Rightarrow}{\not\Leftarrow} A \cap B = \phi$

Lma. (Regularity of metric outer measure)

u^* metric outer measure

$A \subseteq B$, B open

Let $A_n = \left\{ x \in A:\ \rho(x, B^c) \geq \frac{1}{n} \right\}$ for $n \geq 1$.



Then $\lim\limits_{n\to\infty} u^*(A_n) = u^*(A)$

Note: In general, u^* not conti. (unlike finite measure)

Pf.: "$\leq$":

$\therefore u^*(A_n) \leq u^*(A) \quad \forall n$

$\therefore \overline{\lim\limits_{n\to\infty}}\ u^*(A_n) \leq u^*(A)$

"$\geq$":

$\because A_n \uparrow \bigcup\limits_n A_n \subseteq A$

Conversely, $\forall y \in A \subseteq B$, $\exists N_\varepsilon(y) \subseteq B \quad (\because B$ open$)$

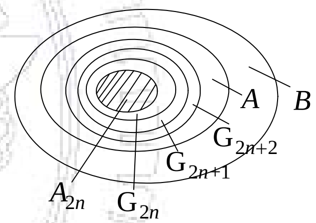
$\Rightarrow \rho(y, B^c) \geq \varepsilon$

Let $n > \frac{1}{\varepsilon}$

$\Rightarrow y \in A_n \qquad (\because \rho(y, B^c) \geq \varepsilon > \frac{1}{n})$

$\Rightarrow y \in \bigcup\limits_n A_n$

$\therefore A_n \uparrow A$



Let $G_n = A_{n+1} \setminus A_n$ for $n \geq 1$.

$\therefore A = A_{2n} \cup (\bigcup\limits_{k=n}^{\infty} G_{2k}) \cup (\bigcup\limits_{k=n}^{\infty} G_{2k+1})$ (disj. union not applicable)

$\Rightarrow u^*(A) \leq u^*(A_{2n}) + \sum\limits_{k=n}^{\infty} u^*(G_{2k}) + \sum\limits_{k=n}^{\infty} u^*(G_{2k+1})$

Check: $\sum\limits_{k=1}^{\infty} u^*(G_{2k})$ & $\sum\limits_{k=1}^{\infty} \mu^*(G_{2k+1})$ converge

$(\Rightarrow u^*(A) \leq \underline{\lim}\ u^*(A_{2n}) \ \Rightarrow \lim u^*(A_n) = \mu^*(A))$

Check: $\sum\limits_{k=1}^{\infty} u^*(G_{2k})$ converges.

$$\because x \in G_{2k} \Rightarrow x \in A_{2k+1} \Rightarrow \rho(x, B^c) \geq \frac{1}{2k+1}$$

$$y \in G_{2k+2} \Rightarrow y \notin A_{2k+2} \Rightarrow \rho(x, B^c) < \frac{1}{2k+2}$$

$$\therefore \rho(x, y) \geq \rho(x, B^c) - \rho(y, B^c) \geq \frac{1}{2k+1} - \frac{1}{2k+2} > 0$$

$$\Rightarrow \rho(G_{2k}, G_{2k+2}) \geq \frac{1}{2k+1} - \frac{1}{2k+2} > 0$$

$$\because A \supseteq A_{2n} \supseteq \bigcup_{k=1}^{n\text{-}1} G_{2k}$$

$$\Rightarrow u^*(A) \geq u^*(A_{2n}) \geq u^*(\bigcup_{k=1}^{n\text{-}1} G_{2k}) = \sum_{k=1}^{n\text{-}1} u^*(G_{2k}).$$

$$\Rightarrow u^*(A) \geq \overline{\lim}\ u^*(A_{2n}) \geq \underline{\lim}\ u^*(A_{2n}) \geq \sum_{k} u^*(G_{2k})$$

(i) $\sum_{k} u^*(G_{2k})$ converges

or (ii) $\sum_{k} u^*(G_{2k})$ diverges $\Rightarrow u^*(A) = \lim u^*(A_{2n}) = \infty$