Class 60

Sec. 4.13. Adjoint operators

Def. X, Y normed spaces

$T : X \to Y$ operator (bdd, linear)

Let $T^* : Y^* \to X^*$ be $\ni$ $\left(T^* y^*\right)(x) = y^*(Tx)$ $\forall\, y^* \in Y^*, x \in X$

(Note: In Hilbert spaces, $\left\langle T^* y^*, x\right\rangle = \left\langle y^*, Tx\right\rangle$)

Ex. $T = \left[a_{ij}\right] \Rightarrow T^* = \left[\overline{a_{ji}}\right]$

Note: (1). $T^* y^* \in X^*$

$$\begin{bmatrix} \text{Reason: } \left(T^* y^*\right)(a x_1 + b x_2) = a\left(T^* y^*\right)(x_1) + b\left(T^* y^*\right)(x_2) \\ \left|\left(T^* y^*\right)(x)\right| = \left|y^*(Tx)\right| \le \left\|y^*\right\| \cdot \|T\| \cdot \|x\|. \\ \Rightarrow \left\|T^* y^*\right\| \le \left\|y^*\right\| \cdot \|T\| \end{bmatrix}$$

(2). $T^* : Y^* \to X^*$ (bdd, linear) operator.

$$\begin{bmatrix} \text{Reason: } T^*\left(a y_1^* + b y_2^*\right) = a T^* y_1^* + b T^* y_2^* \\ \quad \& \ \left\|T^*\right\| \le \|T\| \text{ (from (1)).} \end{bmatrix}$$

Prop. (1) $T \mapsto T^* : B(X, Y) \to B\left(Y^*, X^*\right)$: linear & contractive (i.e., $\left\|T^*\right\| \le \|T\|$)

Reason: $\left(aT_1 + bT_2\right)^* = aT_1^* + bT_2^*$

& $\left\|T^*\right\| \le \|T\|$ (from (1))

Note. (4) $\Rightarrow \left\|T^*\right\| = \|T\|$

(2) X, Y, Z normed spaces

$T \in B(X, Y), S \in B(Y, Z) \Rightarrow ST \in B(X, Z)$

$T^* \in B\left(Y^*, X^*\right), S^* \in B\left(Z^*, Y^*\right) \Rightarrow T^* S^* \in B\left(Z^*, X^*\right)$

Then $(ST)^* = T^* S^*$

Pf: Routine check.

(3) $I : X \to X$ identity

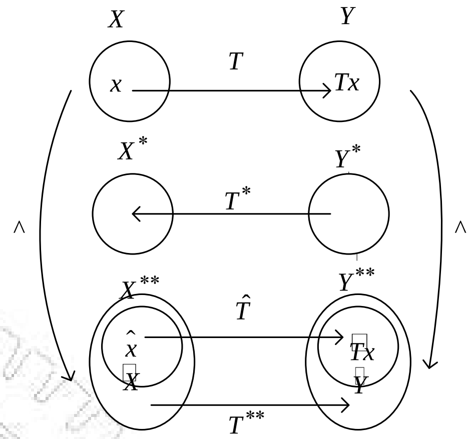
Then $\left(I_X^*\right) = I_{X^*}$

Pf: $\left(I^* y^*\right)(x) = y^*(Ix) = y^*(x)\ \ \forall x \in X$

$\Rightarrow I^* y^* = y^*\ \ \forall y^* \in X^*$

$\Rightarrow I^* = I$ on X^*

$\boxed{\left(T^* y^*\right) x = y^*(Tx)}$



$\hat{T}\hat{x} = \hat{T}x\ \ \forall x \in X$

$\hat{x}\left(x^*\right) = x^*(x)\ \ \forall x \in X, x^* \in X^*$

(4) $T^{**} : X^{**} \to Y^{**}$ is an extension of $\hat{T}$ on $\hat{X}$

Pf.: Let $\hat{x} \in \hat{X}$

Check: $T^{**}(\hat{x}) = \hat{T}(\hat{x}) \in Y^{**}$

Let $y^* \in Y^*$

Check: $\left(T^{**}(\hat{x})\right)\left(y^*\right) = \hat{T}(\hat{x})\left(y^*\right)$

$\|$(def. of T^{**}) $\quad \|$(def. of $\hat{T}$)

$\hat{x}\left(T^* y^*\right) \qquad \left(\hat{T}x\right)\left(y^*\right)$

$\|$(def. of $\hat{x}$) $\quad \|$(def. of $\hat{T}x$)

$\left(T^* y^*\right)(x) \qquad y^*(Tx)$

$\|$(def. of T^*)

$y^*(Tx)$

(5) $\left\|T^*\right\| = \|T\|$

Pf.: $\|T\| = \left\|\hat{T}\right\| \underset{\uparrow}{\le} \left\|T^{**}\right\| \underset{\uparrow}{\le} \left\|T^*\right\|$

(4) (1)

$\Rightarrow \left\|T^*\right\| = \|T\|$

(6) X reflexive $\Rightarrow T^{**} = \hat{T}$

(7) T invertible in $B(X,Y) \Leftrightarrow T^*$ invertible in $B\left(Y^*, X^*\right)$ if X Banach space.

Moreover, $\left(T^{-1}\right)^* = \left(T^*\right)^{-1}$

Pf.: "$\Rightarrow$"

$\therefore TT^{-1} = I_Y$ & $T^{-1}T = I_X$

(2) $\Rightarrow \left(T^{-1}\right)^* T^* = \left(I_Y\right)^* = I_{Y^*}$ & $T^*\left(T^{-1}\right)^* = \left(I_X\right)^* = I_{X^*}$

$\Rightarrow T^*$ invertible with inverse $\left(T^{-1}\right)^*$.

"$\Leftarrow$"

$\because$ T^* invertible in $B\left(Y^*, X^*\right)$

$\Rightarrow T^{**}$ invertible in $B\left(X^{**}, Y^{**}\right)$

& $\hat{T}$ 1-1

> Note: In general, restriction may not be onto.
> Ex. $f : \mathbb{Z} \to \mathbb{Z} \ni f(n) = n+1$ 1-1 & onto.
> Then $f|\mathbb{N}$ is 1-1, but not onto.

Check: $\hat{T} : \hat{X} \to \hat{Y}$ onto.

Assume $\hat{T}\left(\hat{X}\right) \neq \hat{Y}$

Check: $\overline{\hat{T}\left(\hat{X}\right)} \neq \hat{Y}$

Reason: $\because \overline{\hat{T}\left(\hat{X}\right)} = \overline{T^{**}\left(\hat{X}\right)} = T^{**}\left(\hat{X}\right) = \hat{T}\left(\hat{X}\right) \neq \hat{Y}$

> Reason: T^{**} invertible & $\hat{X}$ closed in X^{**} ($\because X$ Banach space)
> $\Rightarrow T^{**}\left(\hat{X}\right)$ closed.
>
> > Pf: $T^{**}\hat{x}_n \to y$, where $\hat{x}_n \in \hat{X}$
> > $\Rightarrow \hat{x}_n \to {T^{**}}^{-1} y \in \hat{X}$ ($\because {T^{**}}^{-1}$ bdd by open mapping thm)
> > $\Rightarrow y = T^{**}\left({T^{**}}^{-1} y\right) \in T^{**}\hat{X}$

$\Rightarrow \overline{TX} \neq Y$

Hahn-Banach $\Rightarrow \exists\, y^* \in Y^* \ni y^* \neq 0$ & $y^*\left(\overline{TX}\right) = \{0\}$

$\Downarrow$

$\left(T^* y^*\right)(X) = \{0\}$

i.e., $T^* y^* = 0$

$\Rightarrow y^* = 0 \to\leftarrow$

($\because T^*$ inv.)

$\therefore \hat{T}$ invertible

$\therefore T$ invertible

X normed space, $A \subseteq X$ subset

Def. $A^{\perp} = \left\{x^* \in X^* : x^*(x) = 0 \ \forall\, x \in A\right\}$

(orthogonal complement of A)

$B \subseteq X^*$ subset

Def. $B^{\perp}=\left\{x\in X: x^*(x)=0\ \forall\ x^*\in B\right\}$

(orthogonal complement of B)

Properties:

(1) $A^{\perp}$ closed subspace of X^*

$B^{\perp}$ closed subspace of X

(2) $X^{\perp}=\{0\}$: trivial

$X^{*\perp}=\{0\}$: Hahn-Banach Thm.

(3) $\underset{\substack{\uparrow \\ \text{in } X}}{\{0\}}^{\perp}=X^*$: trivial

(4) $\underset{\substack{\uparrow \\ \text{in } X^*}}{\{0\}}^{\perp}=X$: trivial

(5) $A^{\perp\perp}=\overline{spanA}$ (Ex)

(6) $B^{\perp\perp}=\overline{spanB}$ (Ex)

(7) $T\in B(X,Y)\Rightarrow\overline{\text{ran}T}=\ker T^{*\perp}$

Note: $T^*:Y^*\to X^*$

$\therefore \ker T^*\subseteq Y^*$

$\therefore \left(\ker T^*\right)^{\perp}\subseteq Y.$

Pf.: "$\subseteq$":

Let $y\in\overline{\text{ran}T}$

Then $\exists\ x_n\in X \ni Tx_n\to$ y

Let $y^*\in\ker T^*$, i.e., $T^*y^*=0$

Check: $y^*(y)=0$

$\uparrow$

$y^*(Tx_n)=\left(T^*y^*\right)(x_n)=0$

"$\supseteq$":

Let $y\notin\overline{\text{ran}T}$

Hahn-Banach Thm $\Rightarrow \exists\, y^* \in Y^* \ni y^*(y) \neq 0$ & $y^*\left(\overline{\operatorname{ran} T}\right) = 0$

$$\Downarrow$$

$$\therefore y^*(Tx) = 0 \ \forall x \in X$$

$$\|$$

$$\left(T^* y^*\right)(x)$$

$$\Rightarrow T^* y^* = 0$$

$$\Rightarrow y^* \in \ker T^*$$

$$\Rightarrow y \notin \ker T^{*\perp}$$